Weyl and Heat asymptotics.

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Question (1)

Are there "two–term" ("second-term") Weyl asymptotics for the isotropic stable processes (fractional Laplacian of order $0 < \alpha < 2$), and even perhaps for other subordinations of Brownian motion such a relativistic Brownian motion, in domains of Euclidean space under Dirichlet boundary conditions? Answer: Unknown as of now

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Question (4)

Question the questions: *Why should we be interested in these questions?*

- $D \subset \mathbb{R}^d$, $d \geq 1$, $|D| = Vol(D)$, $|\partial D| = Area(\partial D)$.

$$\begin{cases} \Delta\varphi_k(x) = -\lambda_k\varphi_k(x), & x \in D \\ \varphi_k(x) = 0, & x \in \partial D \end{cases}$$

has eigenvalues satisfying:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \lambda_k \to \infty.$$

Example

$D = [-1, 1]$ in the real line has eigenfunctions $\varphi_k(x) = \sin(k\pi x)$ and eigenvalues $\lambda_k = k^2\pi^2$.

Example

For square in plane $D = [-1, 1] \times [-1, 1]$, eigenfunctions are products of $sin(k\pi x$ and $\sin(n\pi x)$ and eigenvalues are sums $k^2\pi^2 + n^2\pi^2$, etc.

Example

For the disc in the plane or balls in $\mathbb{R}^d$, eigenfunctions are Bessel functions and eigenvalues are "their" roots but even this case is already complicated.

A Celebrated Theorem of Herman Weyl, know as "Weyl's Law":

Definition

The counting function: $N_D(\lambda) = card\{\lambda_k | \lambda_k < \lambda\}$=number of eigenvalues not exceeding λ.

Theorem (Weyl's Law, 1912:)

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N_D(\lambda) = \frac{|D|}{(4\pi)^{d/2}\Gamma(d/2+1)} = C_d|D|$$

Some History

- Weyl's theorem had been conjectured by Hendrik Antoon Lorentz (1902 Nobel Prize) in October 1910 in his "Old and new problems in physics" lectures at Göttingen.
- Lorentz gave a series of six lectures, the conjecture was stated at the end of the 4th lecture.
- Göttingen had an endowed prize (the Paul Wolfskehl prize) for proving, or disproving, Fermat's last theorem. The donor stipulated that as long as the prize was not awarded, the proceeds from the principal should be used to invite an eminent scientists to deliver a series of lectures. Other eminent scientist that delivered the Wolfskehl Lectures included Poincaré, Einstein, Planck and Bohr.

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- **Note:The prize was awarded to Andrew Wiles in 1997.**

Quoting from M. Kac, celebrated “Can one hear the shape of a drum?":

“Hilbert predicted that the theorem would not be proved in his life time. Well, he was wrong by many, many years (he died in 1943). For less than two years later Hermann Weyl, who was present at the Lorentz’ lecture and whose interest was aroused by the problem, proved the theorem in question. Weyl used in a masterly way the theory of integral equations, which his teacher Hilbert developed only a few years before, and his proof was a crowning achievement of this beautiful theory."

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Weyl wrote three papers on this topic (including other boundary value problems such as the elastic vibrations of a homogeneous body) around this time. He also made the following conjecture.

Conjecture (Weyl 1913)

$$N_D(\lambda) = C_d|D|\lambda^{d/2} - C_d'|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}), \quad as \quad \lambda \to \infty,$$

Theorem (Hörmander 1968, Seeley 1978)

$$N_D(\lambda) = C_d|D|\lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad as \quad \lambda \to \infty.$$

Theorem (Ivri (1980), Melrose (1980): Conjecture is true under)

$\partial D \in C^\infty$ and "measure of all periodic geodesic billiards is zero".

In "Ramifications, old and new, of the eigenvalue problem" Weyl (1950): "I feel that these informations about the proper oscillations of a membrane, valuable as they are still incomplete. I have certain conjectures on what a complete analysis of their asymptotic behavior should aim at; but since for more than 35 years I have made no serious attempt to prove them, I think I had better keep them to myself."

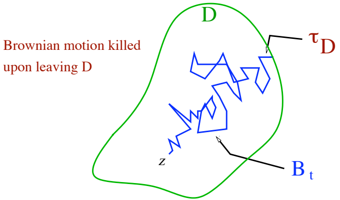
1. **M. Kac early 50's proved that Weyl's one–term asymptotics can be obtained from the heat kernel ("hence" from Brownian motion) and tauberian theorem.**
2. **Kac's method became (and is) the standard method for first order asymptotics. It has been used in many settings. Blumenthal and Getoor used it in 1959 to obtain a Weyl law for stable (and other) processes.**
3. **Kac's method: The behavior of the counting function $N_D(\lambda)$ as $\lambda \to \infty$ follows from the behavior of the trace (partition function) $Z_t(D)$ as $t \to \infty$ of the "heat kernel" ("heat semigroup") associated with the corresponding PDE–in the above case the Laplacia.**

A couple of references

1. For historical account: "Mathematical Analysis of Evolution, information and complexity."

   Chapter titled: "Weyl's Laws: Spectral Properties of the Laplacian in Mathematics and Physics" Wolfgang Arendt, Robin Nittka, Wolfgang Peter, Frank Steiner

2. For many applications of Ivrii and Melrose: The asymptotic distribution eigenvalues of PDE's, Safarov and Vassiliev, AMS Translation monograph, Vol 155–(1996).



$$Z_D(t) = \sum_{j=0}^{\infty} e^{-t\lambda_j} = \int_D p_t^D(x,x)dx = (\text{Laplace transform of } N_d(\lambda))$$

where $P_t^D(x,x)$ =**"the probability that the Brownian particle makes a round trip from x back to x without leaving the domain D."**

$$p_t^D(x,x) \quad = \quad \frac{1}{(4\pi t)^{d/2}} P_x\{\tau_D > t | \, B_t = x\} = p_t(0) P_x\{\tau_D > t | \, B_t = x\}$$

with

$$p_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}}$$

Theorem (M. Kac 1951, perhaps earlier)

$$\lim_{t\downarrow 0} t^{d/2} Z_D(t) = \frac{|D|}{(4\pi)^{d/2}} = p_1(0)\,|D|$$

The Karamata tauberian theorem

Suppose μ is a measure on $[0,\infty)$ with

$$\lim_{t\to 0} t^{\gamma} \int_0^{\infty} e^{-t\lambda} d\mu(\lambda) = A, \quad \gamma > 0.$$

Then

$$\lim_{a\to\infty} a^{-\gamma} \mu[0,a) = \frac{A}{\Gamma(\gamma+1)}$$

Corollary (Kac $\Rightarrow$ Weyl's asymptotics)

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N_D(\lambda) = \frac{p_1(0)\,|D|}{\Gamma(d/2+1)}$$

Theorem (Minakshiusundaram '53–heat invariance)

$D \subset \mathbb{R}^d$, $\partial D \in C^\infty$. Then

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad t \downarrow 0$$

More, the function $Z_D(t)$ has an "expansion" in t for small t.

Theorem (J. Brossard and R. Carmona 1986)

$D \subset \mathbb{R}^d$ is bounded with C^1 boundary. Then

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad t \downarrow 0$$

Remark

For domains as in Ivri-Melrose, Brossard-Carmona also follows by integrating from the asymptotics of $N_D(\lambda)$. That is, from asymptotics of the measure at infinity one gets asymptotics of its Laplace transform at zero–But not the other way around.

Definition (R-Smooth or $C^{1,1}$ domains)

$D \subset \mathbb{R}^d$ is *R-smooth* if $\forall\, x_0 \in \partial D$ there are two open balls B_1 and B_2 with radii R such that

$$B_1 \subset D, \quad B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D) \quad \text{and} \quad \partial B_1 \cap \partial B_2 = x_0.$$

Theorem (M. van den Berg 1987. Uniform in t)

Let ∂D be R–smooth. Then for all $t > 0$,

$$\left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2}\, |\partial D| \right) \right| \leq \frac{d^4}{\pi^{d/2}} \frac{|D|\, t}{t^{d/2} R^2}.$$

Using the celebrated McKean–Singer (1967) Theorem

$$\lim_{t \to 0} \left\{ Z_D(t) - (4\pi t)^{-1} \left(|D| - \frac{\sqrt{\pi t}}{2}\, |\partial D| \right) \right\} = \frac{(1-r)}{6},$$

$D \subset \mathbb{R}^2$, r number of holes in D, van den Berg showed that his bound is best possible in both R and t.

In his 1966 "Can one hear the shape of a drum," Kac derived the second-term asymptotic for $Z_D(t)$ for polygons, i.e., some domains with corners.

Theorem (R. Brown 1993)

If $D \subset \mathbb{R}^d$ is bounded with Lipschitz boundary, then

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad t \downarrow 0$$

Also for Neumann conditions with $(-) \to (+)$ in the second term.

Remark

Proved at the "right time" given the development of harmonic analysis and elliptic boundary value problems in Lipschitz domains of the 80's and 90's.

Remark

Similar results hold with Robin boundary conditions: $\Delta\varphi = -\lambda\varphi$ in D and $\frac{\partial\varphi}{\partial\eta} + k\varphi = 0$ in ∂D. (But only for smooth domains and manifolds.)

For Stable processes

Theorem (R.B.–Kulczycki (2008) & R.B.–Kulczycki, Siudeja (2009))

1. *The van den Berg result (two–term uniform in t for R–smooth domains) and*
2. *The Brown (two–term asymptotics for Lipschitz domains)*

both hold for symmetric stable processes.

Remark

Two-term Weyl asymptotics do not follow from this. (As of now no "heat" analysis gives two-term Weyl asymptotics, even for the Laplacian!). So, the question remains open for fractional Laplacians.

Progress may depend on the development of some kind of "Ivri–Melrose machinery" via a "wave" group for stable processes. Such developments will be very interesting, independently of these applications and perhaps will "free" us probabilists from the "chains" of the heat equation.

Lévy Processes

Constructed by **Paul Lévy** in the 30's (shortly after Wiener constructed Brownian motion). Other names: **de Finetti, Kolmogorov, Khintchine, Itô.**

- Rich stochastic processes, generalizing several basic processes in probability: Brownian motion, Poisson processes, stable processes, subordinators, . . .
- Regular enough for interesting analysis and applications. Their paths consist of continuous pieces intermingled with jump discontinuities at random times. Probabilistic and analytic properties studied by many.
- Many Developments in Recent Years:
  - **Applied:** Queueing Theory, Math Finance, Control Theory, Porous Media . . .
  - **Pure:** Investigations on the "fine" potential and spectral theoretic properties for subclasses of Lévy processes. In recent year there have been many techniques developed to study the heat kernels for many (not all) Lévy processes. This is a very active area of research involving now a very large number of people.

Definition

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

- X has independent and stationary increments
- $X_0 = 0$ (with probability 1)
- X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(dx) = (2\pi)^{d/2}\widehat{p}_t(\xi)$$

The Lévy–Khintchine Formula

The characteristic function has the form $\varphi_t(\xi) = e^{-t\rho(\xi)}$, where

$$\rho(\xi) = -ib\cdot\xi + \langle \mathbb{A}\cdot\xi, \xi\rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\xi\cdot x} + i\xi\cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^d$, a non–negative definite symmetric $n\times n$ matrix $\mathbb{A}$ and a Borel measure ν on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty.$$

$\rho(\xi)$ is called the **symbol** of the process or the **characteristic exponent.** The triple $(b, \mathbb{A}, \nu)$ is called the **characteristics of the process.**

Converse also true. Given such a triplet we can construct a Lévy process.

Examples

1. **Standard Brownian motion:**
   With $(0, I, 0)$, I the identity matrix,
   $$X_t = B_t, \qquad \text{Standard Brownian motion}$$

2. **Gaussian Processes, "General Brownian motion":**
   $(0, \mathbb{A}, 0)$, X_t is "generalized" Brownian motion, mean zero, covariance
   $$E(X_s^j X_t^i) = a_{ij} \min(s, t)$$
   X_t has the normal distribution (assume here that $det(\mathbb{A}) > 0$)
   $$\frac{1}{(2\pi t)^{d/2}\sqrt{\det(\mathbb{A})}} \exp\left(-\frac{1}{2t} x \cdot \mathbb{A}^{-1} x\right)$$

3. **"Brownian motion" plus drift:** With $(b, \mathbb{A}, 0)$ get gaussian processes with drift:
   $$X_t = bt + G_t$$

4. **Poisson Process:** The Poisson Process $X_t = \pi_t(\lambda)$ of intensity $\lambda > 0$ is a Lévy process with $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac delta at 1.

$$P\{\pi_t(\lambda) = m\} = \frac{e^{-\lambda t}(\lambda t)^m}{m!}, \; m = 1, 2, \ldots$$

π_t has continuous paths except for jumps of size 1 at the random times

$$\tau_m = \inf\{t > 0 : \pi_t(\lambda) = m\}$$

5. **Compound Poisson Process** Let $Y_1, Y_2, \ldots$ be i.i.d. and independent of the π_t with distribution ν. The process

$$X_t = Y_1 + Y_2 + \cdots + Y_{\pi_t(\lambda)} = S_{\pi_t(\lambda)}$$

is a Lévy process. By independence

$$\begin{aligned} E[e^{i\xi \cdot X_t}] &= \sum_{m=0}^{\infty} P\{\pi_t = m\} \, E[e^{i\xi \cdot S_m}] \\ &= \sum_{m=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^m}{m!} \left(\widehat{\nu}(\xi)\right)^m = e^{-\lambda t(1-\widehat{\nu}(\xi))} \end{aligned}$$

$$\Rightarrow \rho(\xi) = \lambda \int_{\mathbb{R}^d} \left(1 - e^{ix\cdot\xi}\right) \nu(dx)$$

6. **Relativistic Brownian motion** According to quantum mechanics, a particle of mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

where c is speed of light. Then $\rho(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

7. **The rotationally invariant stable processes:** Self–similar processes X_t^α with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

That is,

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t^\alpha}\right) = e^{-t|\xi|^\alpha}$$

$\alpha = 2$, **Brownian motion,** $\alpha = 1$**, Cauchy processes.** Transition probabilities:

$$P_x\{X_t^\alpha \in A\} = \int_A p_t^\alpha(x-y)dy, \qquad \text{any Borel} \qquad A \subset \mathbb{R}^d$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot x}e^{-t|\xi|^\alpha}d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad \alpha = 2, \qquad \textbf{Brownian motion}$$

$$p_t^1(x) = \frac{C_d\, t}{(|x|^2 + t^2)^{\frac{d+1}{2}}}, \qquad \alpha = 1, \qquad \textbf{Cauchy Process}$$

For any $a > 0$, the two processes

$$\{\eta_{(at)}\,;\ t \geq 0\} \quad \text{and} \quad \{a^{1/\alpha}\eta_t\,;\ t \geq 0\},$$

have the same finite dimensional distributions **(self-similarity).**

In the same way, the transition probabilities scale similarly to those for BM:

$$p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha} x)$$

and

$$p_t^\alpha(0) = t^{-d/\alpha} p_1^\alpha(0)$$

The Lévy "free" semigroup and generator

$$T_t f(x) = E[f(X(t))|X_0 = x] = E_0[f(X(t) + x)], \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

$$T_t f(x) = \int_{\mathbb{R}^d} f(x+y) p_t(dy) = p_t * f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{t\rho(\xi)} \widehat{f}(\xi) d\xi$$

Generator:

$$Af(x) = \frac{\partial T_t f(x)}{\partial t}\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\Big(E_x[f(X(t)] - f(x)\Big)$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \rho(\xi) \hat{f}(\xi) d\xi = \text{a pseudo diff operator, in general}$$

From the Lévy–Khintchine formula (and properties of the Fourier transform),

$$Af(x) = \sum_{i=1} b_i \partial_i f(x) + \sum_{i,j} a_{i,j} \partial_i \partial_j f(x) + \int \Big[f(x+y) - f(x) - y \cdot \nabla f(x) \chi_{\{|y|<1\}} \Big] \nu(dy)$$

Examples:

- Standard Brownian motion (running at twice the usual speed):

$$Af(x) = \Delta f(x)$$

- Poisson Process of intensity λ:

$$Af(x) = \lambda \Big[f(x+1) - f(x) \Big]$$

- Rotationally Invariant Stable Processes of order $0 < \alpha < 2$, **Fractional Diffusions:**

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &= A_{\alpha,d} \int \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy \end{aligned}$$

$$Z_D(t) = \sum_{j=0}^{\infty} e^{-t\lambda_j^{\alpha}} = \int_D p_t^{D,\alpha}(x,x)dx$$

$$= p_t^{\alpha}(0)\,|D| - \int_D r_t^D(x,x)dx$$

Lemma

$$\lim_{t\to 0} t^{d/\alpha} \int_D r_t^D(x,x)dx = 0$$

Corollary (For any set of finite volume D)

$$\lim_{t\to 0} t^{d/\alpha} Z_D(t) = p_1^{\alpha}(0)|D|$$

Corollary (Weyl's asymptotics)

$$\lim_{\lambda\to\infty} \lambda^{-d/\alpha} N_D(\lambda) = \frac{p_1^{\alpha}(0)|D|}{\Gamma(d/\alpha+1)}$$

Proved under assumptions on ∂D by Blumenthal and Getoor 1959.

Theorem (For R-smooth domains: R.B. T. Kulczycki 2009)

$$\left| Z_D(t) - \frac{C_1(\alpha,d)\,|D|}{t^{d/\alpha}} + \frac{C_2(\alpha,d)\,|\partial D| t^{1/\alpha}}{t^{d/\alpha}} \right| \le \frac{C_3\,|D| t^{2/\alpha}}{R^2 t^{d/\alpha}}, \quad t > 0.$$

Theorem (For Lipschitz domains: R.B.–T. Kulczycki, B. Siudeja 2010)

$$t^{d/\alpha} Z_D(t) = C_1(\alpha,d)\,|D| - C_2(\alpha,d)\,|\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right), \quad t \downarrow 0$$

$$C_1(\alpha,d) = p_1^{\alpha}(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

$$C_2(\alpha,d) = \int_0^\infty r_1^H(\vec{q},\vec{q})\,dq, \qquad H = \left\{x \in \mathbb{R}^d : x_1 > 0\right\}, \qquad \vec{q} = (q, 0, \ldots 0)$$

Question

Is there a McKean–Singer type result for stable processes/fractional Laplacian?

The Weyl one-term asymptotic gives: For any $0 < \alpha \leq 2$

$$\lambda_k \sim C(\alpha)\frac{k^{\alpha/d}}{|D|^{\alpha/d}}, \qquad k \to \infty,$$

$C(\alpha) = (4\pi)^{\alpha/2}\,[\Gamma(d/2+1)]^{\alpha/d}$

Pólya's Conjectured (1961), $\alpha = 2$.

$$\lambda_k \geq C(2)\frac{k^{2/d}}{|D|^{2/d}}, \qquad \forall \quad k \geq 1.$$

Known for domains that tile the plane, open even for the disk!

Berezin (1972), Li-Yau (1983): For the Laplacian, $\alpha = 2$

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} C(2)\frac{k^{2/d+1}}{|D|^{2/d}}$$

$$\Rightarrow \quad \lambda_k \geq \frac{d}{d+2} C(2)\frac{k^{2/d}}{|D|^{2/d}}$$

S. Yolcu (2010) and S. Yolcu and T. Yolcu (2010): For the α Laplacian

We also have

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+\alpha} C(\alpha) \frac{k^{1+\alpha/d}}{|D|^{\alpha/d}}.$$

and

$$\lambda_k \geq \frac{d}{d+2} C(\alpha) \frac{k^{\alpha/d}}{|D|^{\alpha/d}}$$

Integrating the counting measure $N_D(s)$

$$\sum_{j=1}^{k} \lambda_j = \int_0^{\infty} (k - N_D(s))_+ \, ds,$$

With $\alpha = 2$, the two-term Weyl asymptotic for $N_D(s)$ gives

$$\sum_{j=1}^{k} \lambda_j = \frac{d}{d+2} C(2) \frac{k^{1+\frac{2}{d}}}{|D|^{2/d}} + \tilde{C}(d) \frac{|\partial D| k^{1+\frac{1}{d}}}{|D|^{1+\frac{1}{d}}} + o(k^{1+\frac{1}{d}}), \quad k \to \infty$$

R. Frank and G. Leader 2010

This holds for all $0 \leq \alpha \leq 2$, even without the two-term Weyl asymptotics. The proof comes from studying the trace of perturbations of "heat" semigroup associated with the operator $\Delta^{\alpha/2}$. It requires less smoothness than the two-term Weyl formula.

Thank you!