The fundamental gap problem

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With special appearances by

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<u>Outline</u>

- Statement of Problem
- First bounds: I. Singer–B.Wang–S.T.Yau–S.S.T.Yau
- Results of Ashbaugh-Benguria, Lavine
- Spectral gaps for probability measures with log–concave densities: Bobkov, Smits, . . .
- **Robert's presentation**
- A more general probabilistic question
- Results for planar symmetric domains
- Multiple Integrals: the heart of the matter
- **Pedro's presentation**
- Problems on gaps and related questions. (Please also see the general list of open problems from the Workshop)
- **Please Note:** The references given are to those provided at the meeting and also available at this site.

Convex Domains of finite diameter

Conjecture on Dirichlet Spectral gaps of Schrödinger operators

$H = -\Delta + V$ with Dirichlet conditions in the bounded convex domain $D \subset \mathbb{R}^n$ of finite diameter d_D, $V \geq 0$ is bounded and convex in D. We have eigenvalues

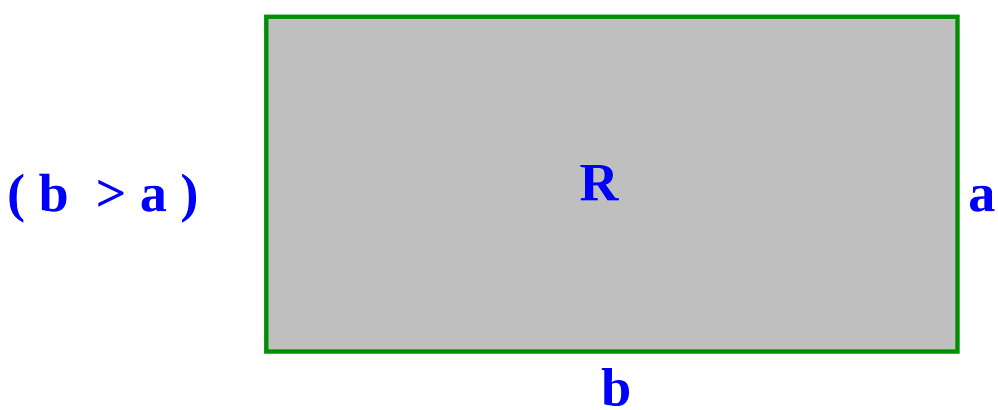
$$0 < \lambda_1(D, V) < \lambda_2(D, V) \leq \lambda_3(D, V) \ldots.$$

Conjecture (M. van den Berg 1983, Ashbaugh–Benguria 1987, and Problem #44 in Yau's 1990 "open problems in geometry")

$$gap(D, V) = \lambda_2(D, V) - \lambda_1(D, V) > \frac{3\pi^2}{d_D^2}$$

with the lower bound approached when $V = 0$ and the domain becomes a thin rectangular box.

False for nonconvex domains even with $V = 0$.



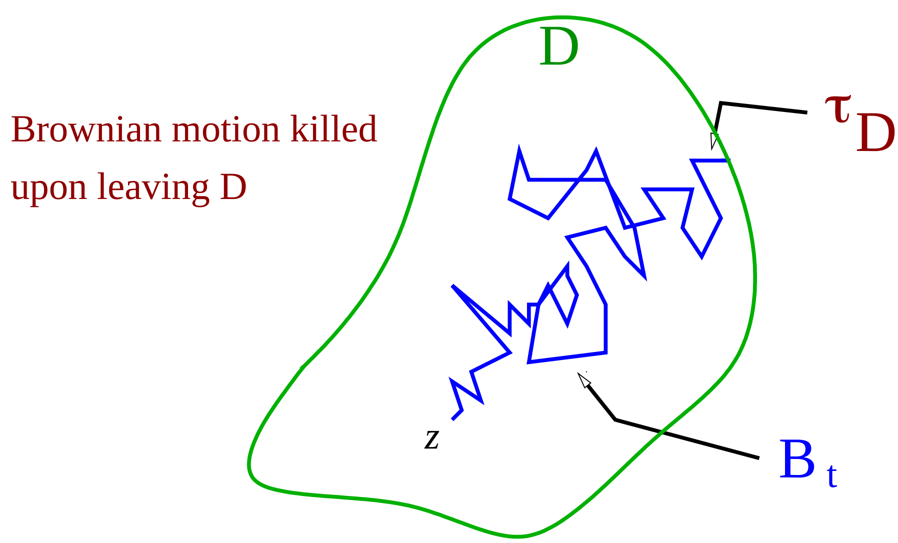
$$\lambda_2(R) - \lambda_1(R) = \left(\frac{4\pi^2}{b^2} + \frac{\pi^2}{a^2}\right) - \left(\frac{\pi^2}{b^2} + \frac{\pi^2}{a^2}\right) = \frac{3\pi^2}{b^2},$$

Heat kernel, eigenfunctions, Brownian motion

$$
\begin{aligned}
K_t(z,w) &= \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(z)\varphi_j(w) \\
&= G_t(z,w)E_z\{e^{-\int_0^t V(B_s)ds}; \tau_D > t \mid B_t = w\},
\end{aligned}
$$

$$
G_t(z,w) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|z-w|^2}{4t}}
$$

For any domain D we let τ_D be the first time that the Brownian motion exits the domain given that it started at a point in D.



$$
K_t(z,w) = G_t(z,w)P_z\{\tau_D > t \mid B_t = w\}, \qquad V = 0
$$

- (Davies–Simon (1984), Smits (1996)) There are constants C_1, C_2 such that for all $t \geq 1$,

$$C_1 e^{-gap(D,V)t} \leq \sup_{z,w \in D} \left| \frac{e^{\lambda_1 t} K_t(z,w)}{\varphi_1(z)\varphi_1(w)} - 1 \right| \leq C_2 e^{-gap(D,V)t}$$

- Time to Equilibrium:

$$T_\epsilon(D,V) = \inf\{t > 0 : \sup_{z,w \in D} \left| \frac{e^{\lambda_1 t} K_t(z,w)}{\varphi_1(z)\varphi_1(w)} - 1 \right| \leq \varepsilon\}$$

First General Result

I.Singer–B.Wang–S.T.Yau–S.S.T.Yau (1985) : (Lower bound by Max Principle using the "P-function" techniques of Payne, Philippin, . . . Upper bound with test functions.)

$$\frac{\pi^2}{4d_D^2} \leq \lambda_2(D,V) - \lambda_1(D,V) \leq \frac{n\pi^2}{r_D^2} + \frac{4(M-m)}{n},$$

$$M = \sup_D V, \qquad m = \inf_D V$$

$$r_D = \quad \text{inradius of D}$$

No convexity needed for Upper Estimate

Show (key inequality):

$$|\nabla u|^2 + (\lambda_2 - \lambda_1)(\mu - u)^2 \le (\lambda_2 - \lambda_1) \sup_D (\mu - u)^2$$

where

$$u = \varphi_2/\varphi_1, \ \mu \ge \sup_D u$$

This gives:

$$|\nabla u|^2 \le (\lambda_2 - \lambda_1)\{\sup_D (\mu - u)^2 - (\mu - u)^2\}$$

or

$$|\nabla u|^2 \le (\lambda_2 - \lambda_1)\{(\sup_D u - \inf_D u)^2 - (\sup_D u - u)^2\}$$

or with $A = \sup u - \inf u$ and $W = \sup u - u$,

$$\sqrt{\lambda_2 - \lambda_1} \ge \frac{|\nabla W|}{\sqrt{A^2 - W^2}}$$

With $u(q_1) = \sup u$, $u(q_2) = \inf u$. Integrate along a segment from $q_1 \in D$ to $q_2 \in D$ to get:

$$\int_{\sup u}^{\inf u} \frac{|\nabla u| ds}{\sqrt{(\sup_D u - \inf_D u)^2 - (\sup_D u - u)^2}} \le \int_{q_1}^{q_2} \sqrt{\lambda_2 - \lambda_1} ds$$

$$\int_0^A \frac{|dW|}{\sqrt{A^2 - W^2}} \le \int_{q_1}^{q_2} \sqrt{\lambda_2 - \lambda_1} ds$$

$$\frac{\pi}{2} \le (\sqrt{\lambda_2 - \lambda_1})(length\ of segment[q_1, q_2]) \le \sqrt{\lambda_2 - \lambda_1} d_D$$

Used in the proof of key inequality: with

$$L = \Delta + 2\frac{\nabla\varphi_1}{\varphi_1}\cdot\nabla$$

have:

$$L\left(\frac{\varphi_n}{\varphi_1}\right) = -(\lambda_n - \lambda_1)\left(\frac{\varphi_n}{\varphi_1}\right)$$

and for D with nice smooth boundary

$$\frac{\partial}{\partial\eta}\left(\frac{\varphi_n}{\varphi_1}\right) = 0, \quad \text{(see SWYY)}$$

$$\lambda_2 - \lambda_1 = \inf\left\{\frac{\int_D |\nabla f|^2\varphi_1^2 dx}{\int_D |f|^2\varphi_1^2 dx}; f\in C^\infty(D), \int_D f\varphi_1^2\,dx = 0\right\}$$

This plus **Brascamp–Lieb:** With V convex and D convex, φ_1 is log–concave.

Yu–Zhong(1986), Ling(1993) improved lower bound to

$$\frac{\pi^2}{d_D^2}$$

by refinement the max principle ("P-function") method.

A sharp upper bound

Take $V = 0$. "similar" argument for $V \geq 0$ by Smits. Payne-Pólya-Weinberger (PPW) Conjecture (Ashbaugh-Benguria 1992):

$$\frac{\lambda_2}{\lambda_1}\Big|_D \leq \frac{\lambda_2}{\lambda_1}\Big|_{Ball} \qquad (= 2.539\ldots,\ n = 2)$$

This is the same as:

$$(\frac{\lambda_2 - \lambda_1}{\lambda_1})\Big|_D \leq (\frac{\lambda_2 - \lambda_1}{\lambda_1})\Big|_{B(0,\,r_D)}$$

which gives

$$(\lambda_2 - \lambda_1)\big|_D \leq \{(\lambda_2 - \lambda_1)\big|_{B(0,\,r_D)}\}\{\frac{\lambda_D}{\lambda_{(B(0,r_D)}}\}$$

$$(\lambda_2 - \lambda_1)\big|_D \leq (\lambda_2 - \lambda_1)\big|_{B(0,\,r_D)}$$

The spectral gap is bounded above by that of the smallest disk contained in the domain

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$$\begin{aligned}(\lambda_2 - \lambda_1)\big|_{D,V} &\leq (\lambda_2 - \lambda_1)\big|_{B(0,\,r_D)} + (\frac{\lambda_2 - \lambda_1}{\lambda_1})\Big|_{Ball}(M - m) \\ &= \frac{\alpha_n}{r_D^2} + \beta_n(M - m)\end{aligned}$$

(See Smits (1996))

Lower bounds on gap (Better than SWYY)

I. The full conjecture, and only known case of the conjecture, was proved by **Richard Lavine** in $\mathbb{R}$.

(*See "The eigenvalue gap for one dimensional convex potentials," Proc. Amer. Math. Soc., 121* (1994), 815–821)

II. When $D = (-b, b)$ and V is symmetric about 0 and increasing in $(0, b)$.

or

III. When D is a disk in $\mathbb{R}^2$ (any $\mathbb{R}^n$), V is radial increasing and $(rV)'' \geq 0$. In particular when V is radial increasing and convex.

(**Ashbaugh and Benguria, 1988, 1989**)

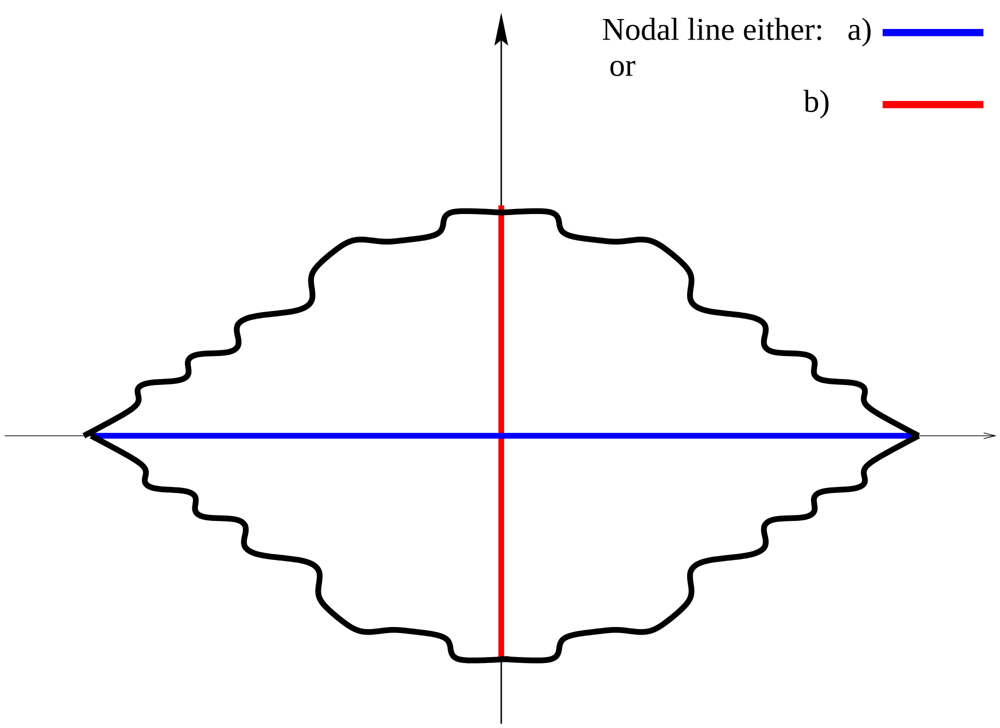
IV. Both II and III also follow from the "multiple integrals" techniques discussed below.

Robert Smits: "Spectral gaps and Rates to equilibrium for diffusions in Convex Domains" (Michigan Math Journal 1996. He uses the variational characterization of $\lambda_2 - \lambda_1$.)

I. For all $D \subset \mathbb{R}^n$ convex and all nonnegative convex potentials V,

$$\frac{\pi^2}{d_D^2} < \lambda_2(D,V) - \lambda_1(D,V).$$

II. If $D \subset \mathbb{R}^2$ is convex and symmetric with respect to the coordinate axes (as in the picture below) and $V = 0$, d_D can be replaced by the length of longest axes of symmetry.



Bobkov (1999): μ a probability measure in $\mathbb{R}^n$ with a log–concave density supported in a set D of diameter d. Its "spectral gap" satisfies:

$$gap(\mu) = \inf\left\{\frac{E|\nabla f|^2}{E|f|^2}; \text{Lips f}, Ef = 0\right\} \geq \frac{(log 2)^2}{d_D^2}$$

- Bobkov gets it from a lower bound on Cheeger's isoperimetric constant: Largest c, called $I_s(\mu)$, such that

$$\mu^+(A) \geq c \min\{\mu(A), 1 - \mu(A)\},$$

$$\mu^+(A) = \liminf_{h\to 0} \frac{\mu(A^h) - \mu(A)}{h}$$

$$A^h = \{x \in \mathbb{R}^n : \exists a \in A, |x - a| < h\}$$

$$gap(\mu) \geq I_s^2(\mu)/4$$

- Bobkov: For every log–concave probability measure μ (it has density which is log-concave)

$$I_s(\mu) \geq \frac{1}{K\|z - z_0\|_{L^2(\mu)}},$$

z_0 is the barycenter of μ, i.e., $z_0 = E(z) = \int_D z d\mu(z)$.

Smits reduces to a one-dimensional Schrödinger operator problem.

- Let ν be the first nonzero eigenvalue for $-\Delta$ with Neumann conditions in the convex D. Then
$$\frac{\pi^2}{d_D^2} \le \nu \le \frac{j_o^2}{d_D^2}$$
Lower bound: Payne-Wienberger (1960), upper bound, S.Y. Chen(1975).

- Smits used:
$$gap(D,V) = \inf \frac{\int_D |\nabla f|^2 \varphi_1^2 dx}{\int_D |f|^2 \varphi_1^2 dx},$$
inf over all $f's$ with integral zero against φ_1^2. Following P–W, Smits shows that
$$\lambda_2 - \lambda_1 \ge \lambda_0,$$

λ_0 is the smallest Dirichlet eigenvalue for
$$-\frac{d^2}{dx^2} + q$$
in the interval $[0, d_D]$, where
$$\begin{aligned} q &= -\{\frac{1}{2}\frac{(p\varphi_1^2)''}{p\varphi_1^2} - \frac{3}{4}\frac{[(p\varphi^2)']^2}{(p\varphi_1^2)^2}\} \\ &= -\frac{1}{2}(\log(p\varphi_1)^2)'' + \frac{1}{4}\frac{[(p\varphi_1^2)']^2}{(p\varphi_1^2)^2} \end{aligned}$$

Smits' Presentation: Robert will give some details

Back to Probability

Let X_t be Brownian motion conditioned to remain forever in D: Diffusion with generator

$$L = \Delta + 2\frac{\nabla \varphi_1}{\varphi_1} \cdot \nabla$$

Set

$$\eta_t = \int_0^t \frac{\varphi_2}{\varphi_1}(X_s)ds, \;\; \sigma = \frac{\sqrt{2}}{\sqrt{\lambda_2 - \lambda_1}}.$$

Then

$$\limsup_{t\to\infty} \frac{\eta_t}{\sqrt{2t \log\log t}} = \sigma \quad \text{a.s.} \quad P_x,$$

and

$$\lim_{t\to\infty} P_x \left\{ \frac{\eta_t}{\sigma\sqrt{t}} > \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty e^{-\frac{r^2}{2}} \, dr.$$

These results follow by applying **Philipp and Stout** "Invariance Principles Techniques" as in Memoirs of AMS, #161, 1975. See Bañuelos (1992) for more.

However, such formulas seem to be completely useless for estimating gaps. Next, we have a more useful probabilistic interpretation of eigenvalues.

Take $V = 0$ (same for nonzero V)

$$\begin{aligned} P_z\{\tau_D > t\} &= \int_D K_t(z,w)dw \\ &= \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(z)\int_D \varphi_j(w)dw \end{aligned}$$

Gives:

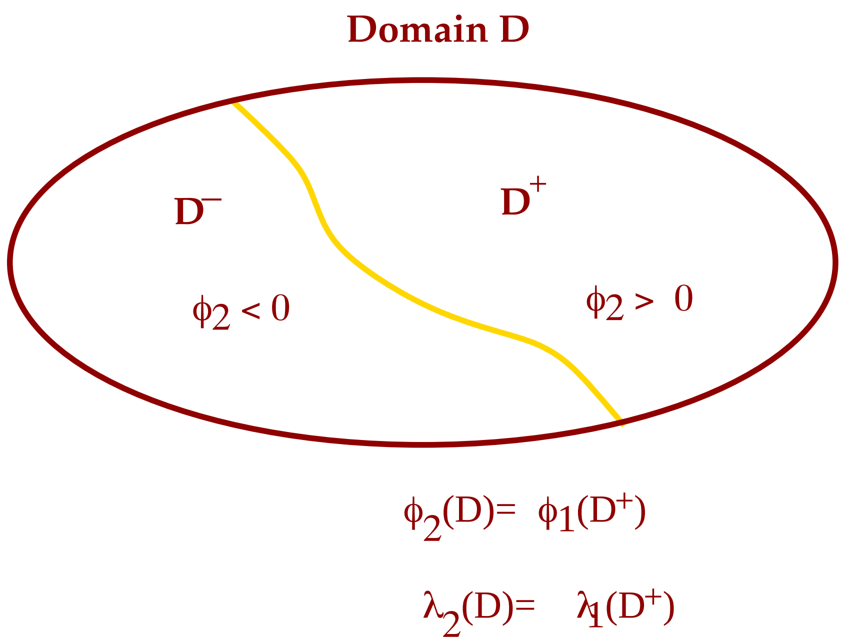
$$\lim_{t\to\infty}\frac{1}{t}\log P_z\{\tau_D > t\} = -\lambda_1(D)$$

Under our assumptions of convexity we have more:

$$\lim_{t\to\infty} e^{\lambda_1(D)t}P_z\{\tau_D > t\} = \varphi_1(z)\int_D \varphi(w)dw,$$

uniformly in $z \in D$.

For the rest of the talk, $D \subset \mathbb{R}^2$ convex. **From A. Melas (1992),** *"On the nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$"* we have the picture:



More General Question (Conjecture): Set

$$I = \left(\frac{-d_D}{2}, \frac{d_D}{2}\right), \quad I^+ = \left(0, \frac{d_D}{2}\right)$$

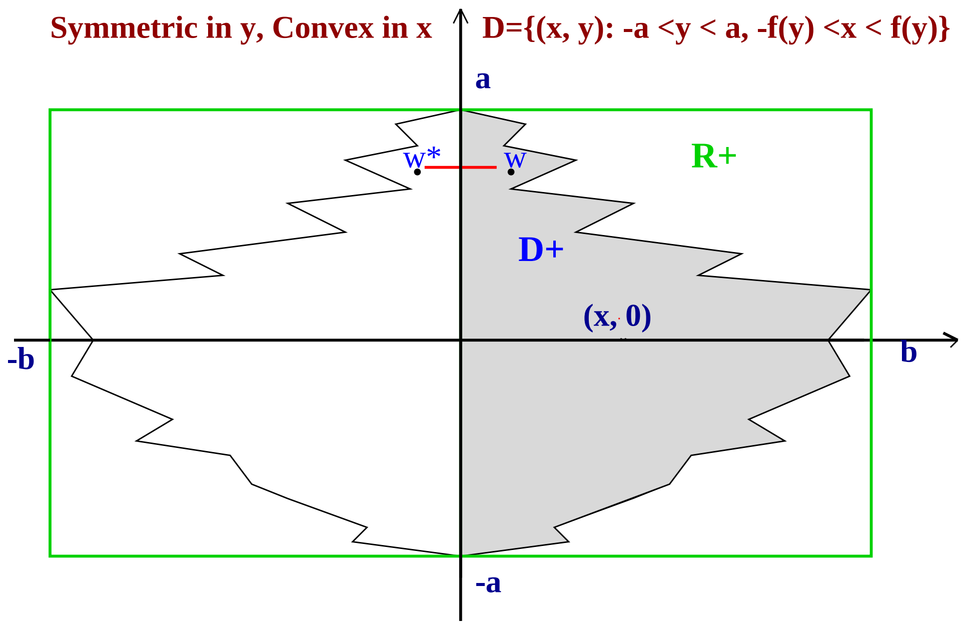
Is there a point $z_0 \in D^+$ and a point $x_0 \in I^+$ such that

$$\frac{P_{z_0}\{\tau_{D^+} > t\}}{P_{z_0}\{\tau_D > t\}} \leq \frac{P_{x_0}\{\tau_{I^+} > t\}}{P_{x_0}\{\tau_I > t\}}?$$

More Ambitious: Is the following trace inequality true?

$$\frac{\int_{D^+} K_t^{D^+}(z,z)dz}{\int_D K_t^D(z,z)dz} \leq \frac{\int_{I^+} K_t^{I^+}(z,z)dz}{\int_I K_t^I(z,z)dz}$$

Both true under symmetry from the following results:



Suppose $V(x,y) \geq 0$ is symmetric in x, increasing in x and D and D^+ are as in the picture.

(Bañuelos–Méndez 1999) For $(x,0) \in D^+$ and all $t > 0$,

$$\frac{\int_{D^+} K^V_{D^+}(t,(0,x),w)dw}{\int_D K^V_D(t,(0,x),w)dw} \leq \frac{\int_0^b K_{(0,b)}(t,x,y)dy}{\int_{-b}^b K_{(-b,b)}(t,x,y)dy}$$

(Dahae You: 2002) For all $t > 0$, the trace inequality holds:

$$\frac{\int_{D^+} K^V_{D^+}(t,w,w)dw}{\int_D K^V_D(t,w,w)dw} \leq \frac{\int_0^b K_{(0,b)}(t,y,y)dy}{\int_{-b}^b K_{(-b,b)}(t,y,y)dy}$$

The "half" interval $(0,b)$ can be replaced by the "right half" rectangle, R^+, and interval $(-b,b)$ by the rectangle R.

You's Theorem Equivalent to: (In terms of Partition Functions): For all $t > 0$,

$$\frac{\sum_{j=1}^{\infty} e^{-t\lambda_j(D^+,V)}}{\sum_{j=1}^{\infty} e^{-t\lambda_j(D,V)}} \leq \frac{\sum_{j=1}^{\infty} e^{-t\lambda_j(I^+)}}{\sum_{j=1}^{\infty} e^{-t\lambda_j(I)}}$$

With $V = 0$

(i) **You equivalent to (Trace inequality for cylinders (see picture below):** For all $t > 0$,

$$\int_{D^+\times(-b,b)} K_t^{D^+\times(-b,b)}(z,z)dz \leq \int_{D\times(0,b)} K_t^{D\times(0,b)}(z,z)dz$$

(ii) **B–Méndez is equivalent to (after taking complements):** For all $t > 0$,

$$P_{x,0)}\{\tau_{R^+} \leq t | \tau_R > t\} \leq P_{(x,0)}\{\tau_{D^+} \leq t | \tau_D > t\}$$

- **In English:** The probability that the Brownian motion hits the segment of symmetry before time t given that it has not yet exited the domain D by time t is larger than the probability for the rectangle.

- Similar ratio inequalities were also proved by B. Davis (2001) from which spectral gap estimates also follow.

Two domains in

3-dimensions:

Right Half

and

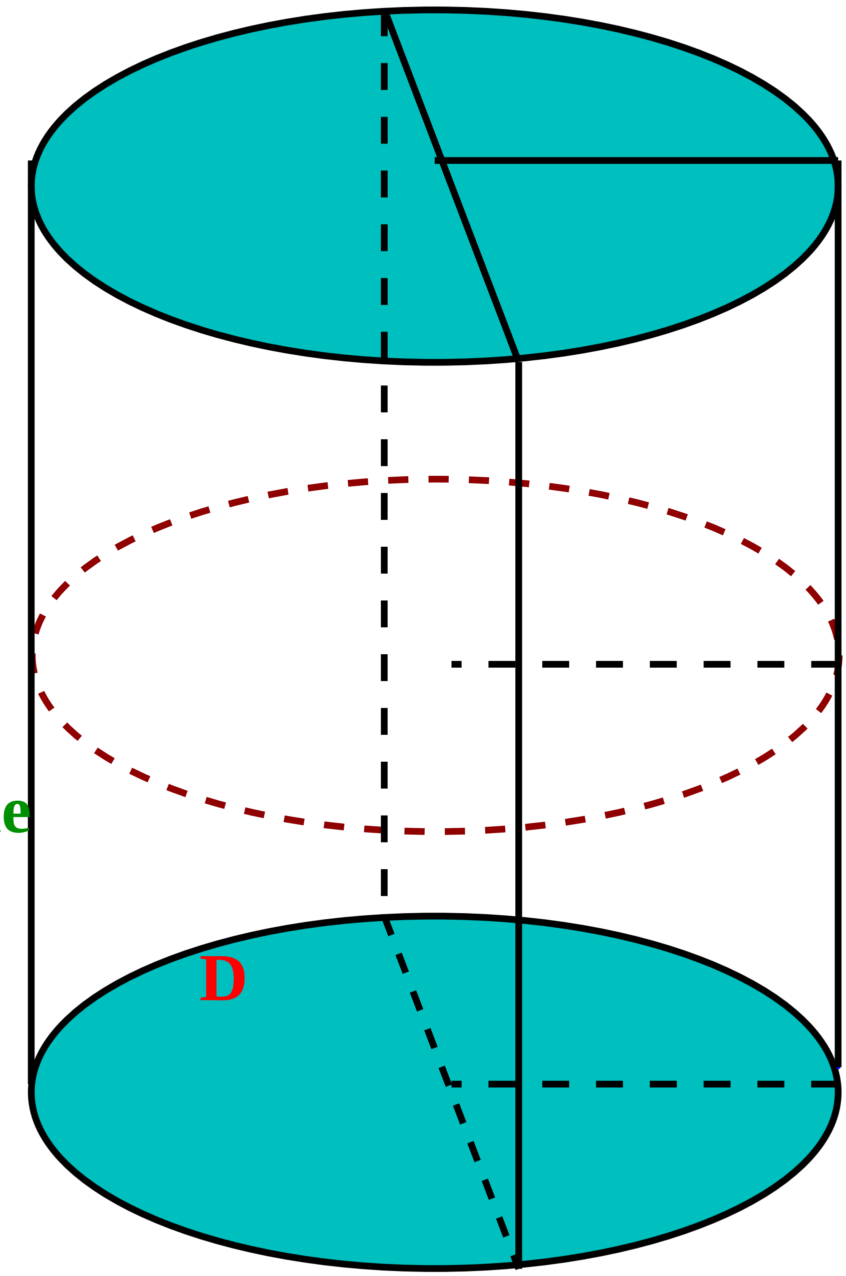
Bottom Half

Which Half has the

smallest Dirichlet

eigenvalue?

Diameter= Height



D- symmetric relative to y

Recall Again:

$$\lim_{t\to\infty} \frac{1}{t} \log \left(\int_D K_D^V(t,(0,x),w)dw \right) = -\lambda_1(D,V)$$

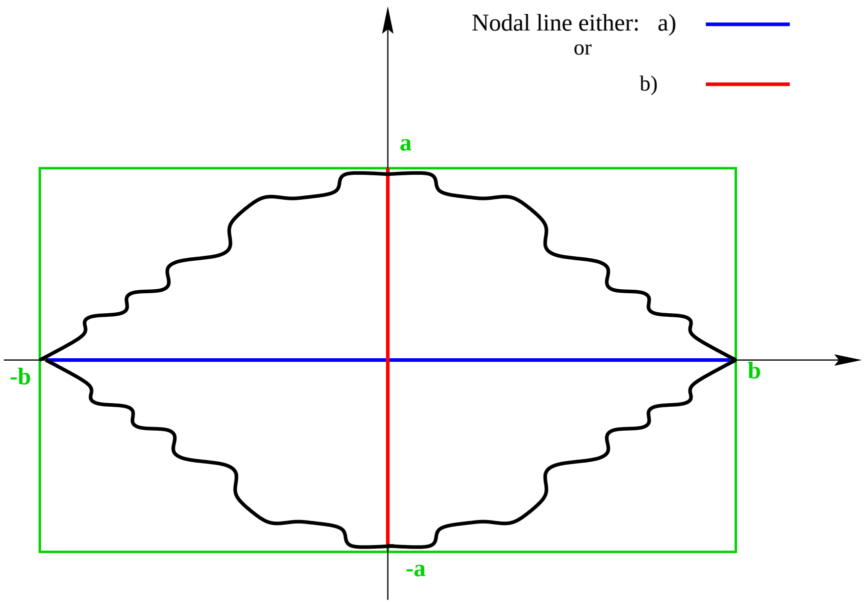
with a similar formula for $D^+, R, R^+, (-b,b), (0,b)$.

We Have:

$$\lambda_1(D^+,V) - \lambda_1(D,V) \geq \lambda_1(0,b) - \lambda_1(-b,b) = \frac{3\pi^2}{b^2}$$

Question. When can we replace $\lambda_1(D^+,V)$ by $\lambda_2(D,V)$?

Answer. At least in the following three situations:



I. **When** $V = 0$ and D symmetric and convex in both axes (Using a result of L. Payne). Proves the conjecture for such D.

II. **When** $D = (-b, b)$ and V is symmetric about 0 and increasing in $(0, b)$. Reproves results of Ashbaugh–Benguria.

III. **When** D is a disk in $\mathbb{R}^2$ (any $\mathbb{R}^n$), V is radial increasing and $(rV)'' \geq 0$. In particular when V is radial increasing and convex. Reproves results of Ashbaugh–Benguria.

Radial Increasing is not enough: There are smooth, positive, bounded and radial increasing potential in the disk $D = B(0, 1) \subset \mathbb{R}^2$ for which λ_2 is simple, its eigenfunction has a closed nodal line and

$$\lambda_1(D^+, V) > \lambda_2(D, V)$$

Bad News!

(See Bañuelos–Méndez (1999).)

Multiple Integrals: the heart of the matter

For any $D \subset \mathbb{R}^n$, any $n \geq 1$,

$$\begin{aligned} P_z\{\tau_D > t\} &= P_z\{B_s \in D;\ \forall s, 0 < s \leq t\} \\ &= \lim_{m\to\infty} P_z\{B_{jt/m} \in D, j = 1, 2, \ldots, m\} \\ &= \lim_{m\to\infty} \int_D \cdots \int_D G_{\frac{t}{m}}(z - z_1) \cdots G_{\frac{t}{m}}(z_m - z_{m-1}) dz_1 \ldots dz_m \end{aligned}$$

$$G_t(z, w) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|z-w|^2}{4t}}$$

Let $z_0 = (x, 0)$. Enough to prove that for every m.

$$\frac{\int_{D^+} \cdots \int_{D^+} G_{\frac{t}{m}}(z_0 - z_1) \cdots G_{\frac{t}{m}}(z_{m-1} - z_m) dz_1 \cdots dz_m}{\int_D \cdots \int_D G_{\frac{t}{m}}(z_0 - z_1) \cdots G_{\frac{t}{m}}(z_{m-1} - z_m) dz_1 \cdots dz_m} \leq$$

$$\frac{\int_0^b \cdots \int_0^b G_{\frac{t}{m}}(x, s_1) \cdots G_{\frac{t}{m}}(s_{m-1} - s_m) ds_1 \cdots ds_m}{\int_{-b}^b \cdots \int_{-b}^b G_{\frac{t}{m}}(x, s_1) \cdots G_{\frac{t}{m}}(s_{m-1} - s_m) ds_1 \cdots ds_m}$$

Pedro Méndez will now give some details

Problems on gaps and related questions

1. **(More general than the gap conjecture when $V = 0$)** Investigate the more general conjectures/questions stated above: Set

$$I = (\frac{-d_D}{2}, \frac{d_D}{2}), \quad I^+ = (0, \frac{d_D}{2})$$

Question: Is there a point $z_0 \in D^+$ (D^+ as in Melas' Theorem) and a point $x_0 \in I^+$ such that for all $t > 0$

$$\frac{P_{z_0}\{\tau_{D^+} > t\}}{P_{z_0}\{\tau_D > t\}} \le \frac{P_{x_0}\{\tau_{I^+} > t\}}{P_{x_0}\{\tau_I > t\}}?$$

More Ambitious: Is the following trace inequality true?

$$\frac{\int_{D^+} K_t^{D^+}(z,z)dz}{\int_D K_t^D(z,z)dz} \le \frac{\int_{I^+} K_t^{I^+}(z,z)dz}{\int_I K_t^I(z,z)dz}, \quad \text{for all } t > 0$$

2. **(General Chegeer isoperimetric inequality)** Can the Chegeer isoperimetric inequality (as in Bobkov above) be used to obtain the known bound of π^2/d^2 for arbitrary convex domains $D \subset \mathbb{R}^n$ and arbitrary nonnegative convex potentials? Such a proof will add more tools to our efforts on the sharp bound. (In the same spirit, find an alternate proof of Lavine's theorem in one dimension and a "P-function" proof of the results of Bañuelos and Méndez–Hernández for symmetric domains.)

3. (**“Hot–Spots” for Brownian motion conditioned to remain forever in a convex domain)** Take $V = 0$ and $D \subset \mathbb{R}^2$ convex. Consider the function $u(x) = \varphi_2(x)/\varphi_1(x)$. **Prove that** (in analogy with the classical “hot–spots” conjecture of Jeff Rauch) u attains its maximum and its minimum on the boundary, and only on the boundary, of D. (Note: For more precise results when the domain is symmetric, please see Bañuelos and Méndez-Hernández, “Hot–Spots for Conditioned Brownian Motion.”)

4. (**Monotonicity of heat kernels in the ball)** Consider the unit ball $B = B(0, 1)$ in $\mathbb{R}^n$. Let $P_t^N(x, y)$ be the Neumann heat kernel for B. A conjecture made by R. Laugesen and C. Morpurgo some years ago (and which surprisingly remains open) asserts that the (radial) function $P_t^N(x, x)$ increases as $|x|$ increases. Consider now the Dirichlet heat kernel for B, $P_t^D(x, y)$, and let φ_1 be the first Dirichlet ground state eigenfunction. Given the discussion above it is natural to conjecture that $P_t^D(x, x)/\varphi_1^2(x)$ increases as $|x|$ increases. Except for the multiplicative factor $e^{\lambda_1 t}$, this is the diagonal of the heat kernel for the Brownian motion conditioned to remain forever in D (also a “Neumann” heat kernel as observed above). We note that it is well known that $P_t^D(x, x)$ decreases as $|x|$ increases. Both of these problems are open even for the disk in the plane.

5. (**Melas for Schrödinger)** Does the Melas (1992) Nodal Line Theorem hold for planar convex domains for $-\Delta +$

V when V is convex? (There are radial increasing potentials in the disk for which this is false, see Bañuelos and Méndez-Hernández, *"Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps."*)

6. **(On the nodal line conjecture)** It is currently unknown whether the Melas Nodal Line Theorem holds for simply connected planar domains. What is known (result of M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili (1997)) is that it does not hold for domains with sufficiently high multiplicity, i.e., domains with too many holes. In 1999, K. Burdzy and W. Werner constructed a counter example to the classical "hot–spots" conjecture which is (in various ways) similar to the Hoffmann-Ostenhof–Hoffmann-Ostenhof–Nadirashvili example. Recently Burdzy (see Burdzy (2005)) constructed a planar domain with one hole where the "hot–spots" conjecture does not hold.

   Question: Is there a planar domain (perhaps even the Burdzy domain) with one hold where the Melas Theorem fails?

7. **(Eigenvalues and eigenfunctions for symmetric stable processes)** Investigate sharp bounds for low eigenvalues when the Brownian motion above is replaced by a symmetric stable Lévy processes of order α, $0 < \alpha < 2$ (the Dirichlet Laplacian is the case $\alpha = 2$). In particular, investigate sharp bounds for λ_1 and for the spectral gap

$\lambda_2 - \lambda_1$. These are interesting questions and completely open even for the interval $(-1, 1)$ in $\mathbb{R}$.

Please Note: This is not the same as simply taking fractions of the Dirichlet Laplacian using the spectral theorem. Doing that yields an operator with the same eigenfunctions as the Dirichlet Laplacian with eigenvalues which are $\alpha/2$ powers of those for the Dirichlet Laplacian. Hence nothing really new nor interesting comes out of such a construction.

Also, investigate the **Brascamp–Lieb concavity** result for the ground state eigenfunctions for these operators.

For more information on the above, and many other related questions and problems, we refer the reader to the papers of Bañuelos, Kulczycki and Méndez-Hernández given in the references. At present there are very few tools available to study the "fine" spectral theoretic properties for these processes and most questions are completely open.

8. **(Extremal domains for the Neumann gap)** Consider the "spectral gap" bound for the Neumann problem of Payne and Wienberger for convex domains of diameter d,

$$\frac{\pi^2}{d^2} \leq \mu.$$

Question: Investigate (motivated by the talk of Professor **Antoine Henrot** at this Workshop) the existence of extremal domains for this inequality.

9. **(The fundamental frequency of a simply connected drum)** Consider a simply connected domain in the plane of finite inradius r. (This quantity is defined as $r = \sup\{d_D(x) : x \in D\}$ where $d_D(x)$ denotes the distance from the point x to the boundary of D.) The following inequality holds: There is a universal constant a such that

$$\frac{a}{r^2} \leq \lambda_1 \leq \frac{j_0^2}{r^2},$$

where $j_0 \approx 2.4048$ is the smallest positive zero of the first Bessel function and a is a universal constant. The upper bound is attained by the disk of radius r.

A problem of considerable interest for many years, which remains open, has been the identification of the best constant a and the existence of extremal domains. For the best bound available for a (0.6197), we refer the reader to Bañuelos and Carroll (1995).

Clearly there are no extremals in the class of bounded domains. But the situation here is more complicated than it first appears given the examples constructed in Bañuelos and Carroll.