L^p–BOUNDS FOR THE BEURLING–AHLFORS TRANSFORM

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ABSTRACT. Let B denote the Beurling-Ahlfors transform defined on $L^p(\mathbb{C})$, $1 < p < \infty$. The celebrated conjecture of T. Iwaniec states that its L^p norm $\|B\|_p = p^* - 1$ where $p^* = \max\{p, \frac{p}{p-1}\}$. In this paper the new upper estimate

$$\|B\|_p \leq 1.575\,(p^* - 1), \quad 1 < p < \infty$$

is found.

1. Introduction

The purpose of this note is to present some improvements to known estimates of the operator norm of the Beurling-Ahlfors transform. This singular integral operator B defined on $L^p(\mathbb{C})$, $1 < p < \infty$, by

$$Bf(z) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}dm(w), \tag{1.1}$$

is of fundamental importance in several areas of mathematics including PDE and quasiconformal mappings. On $L^2(\mathbb{C})$ the operator is given as a Fourier multiplier of $\widehat{Bf}(\xi) = m(\xi)\hat{f}(\xi)$ where

$$m(\xi) = \frac{\overline{\xi}}{\xi}.$$

Thus, B is an isometry on $L^2(\mathbb{C})$ and in particular $\|B\|_2 = 1$. See, [1], [11], [13], [14], [15]. An outstanding open problem of the past 25 years is the computation of its L^p norm for $1 < p < \infty$. In [16], Lehto shows that $\|B\|_p \geq p^* - 1$ and T. Iwaniec conjectures in [13] that

$$\|B\|_p = p^* - 1, \tag{1.2}$$

where $p^* = \max\{p, \frac{p}{p-1}\}$. Up to now, all the techniques which give explicit estimates on the norm of B depend heavily on the work of Burkholder who succeeds in computing best constants of martingale transforms ([4], [6]). In [3], Bañuelos and Wang use the martingale inequalities of Burkholder to prove the preliminary upper bound $\|B\|_p \leq 4(p^* - 1)$. They show that the

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Beurling–Ahlfors transform of a function f can be attained as the projection of martingale transforms of the composition of the harmonic extension of f with Brownian motion; from here Burkholder's estimates take over. Nazarov and Volberg [17] lower the bound to $2(p^*-1)$ using an analytic approach with Bellman functions that ultimately also depends on the martingale inequalities of Burkholder. A different proof of this bound is obtained in [2] using essentially the same proof as the one in [3] but applied to "heat" martingales. Finally, Dragičević and Volberg [12] refine the Bellman function/martingale techniques and make a further observation that gives the following asymptotic result:

$$\|B\|_p \leq \sqrt{2}(p-1)\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos(\theta)|^p d\theta\right)^{-\frac{1}{p}}, \quad 2 \leq p < \infty. \tag{1.3}$$

In this note, we refine the techniques of [3] and [2], utilize certain symmetries in B and prove the following theorem.

Theorem 1.1. *Let $f \in L^p(\mathbb{C};\mathbb{C})$, $2 \leq p < \infty$. Then*

$$\|Bf\|_p \leq \sqrt{2(p^2-p)}\, \|f\|_p. \tag{1.4}$$

If $f : \mathbb{C} \to \mathbb{R}$, then

$$\|Bf\|_p \leq \sqrt{p^2-p}\, \|f\|_p. \tag{1.5}$$

Observe that as in the previous estimates, the starting value of (1.4) when $p = 2$ is 2 whereas as mentioned above, $\|B\|_2 = 1$. Using this, Theorem 1.1 and the Riesz-Thorin interpolation theorem, we will obtain the following general estimate.

Theorem 1.2. *Let $f \in L^p(\mathbb{C};\mathbb{C})$, $1 < p < \infty$. Then*

$$\|Bf\|_p \leq 1.575\,(p^*-1)\, \|f\|_p. \tag{1.6}$$

Remark 1.1. *We note that the estimate from (1.4) $\|B\|_p \leq \sqrt{2(p^2-p)}$, is already asymptotically better than (1.3). To see this, divide both terms by $\sqrt{2}(p-1)$ and raise to the power of p. Let $p \to \infty$. Then the cosine term diverges and the new result converges to $\sqrt{e}$.*

Theorem 1.1 is a consequence of the corresponding martingale Theorem 1.3 below. The advantage of this result in the martingale setting is that it leads to a natural conjecture for best constants for special martingales. Let $X_t = (X_t^1, X_t^2, \ldots, X_t^n)$ and $Y_t = (Y_t^1, Y_t^2, \ldots, Y_t^n)$ be two $\mathbb{R}^n$–valued martingales on the filtration of d–dimensional Brownian motion. We can assume that

$$X_t^j = \int_0^t H_s^j \cdot dB_s,\ Y_t^j = \int_0^t K_s^j \cdot dB_s,$$

where B_t is d–dimensional Brownian motion and H_s and K_s are $\mathbb{R}^d$–valued processes adapted to its filtration. As usual,

$$\langle X\rangle_t = \sum_{j=1}^{n}\langle X^j\rangle_t = \sum_{j=1}^{n}\int_0^t |H_s^j|^2\,ds$$

denotes the quadratic variation process of X_t with a similar definition for $\langle Y\rangle_t$. Also,

$$\langle X^i, Y^j\rangle_t = \int_0^t H_s^i\cdot K_s^j\,ds$$

denotes the covariation process. We say that Y is differentially subordinate to X if $d\langle Y\rangle_t = \sum_{j=1}^n |K_t^j|^2 \le \sum_{j=1}^n |H_t^j|^2 = d\langle X\rangle_t$, a.e. for all $t>0$. Our main martingale result is the following.

Theorem 1.3. *Let X and Y be two $\mathbb{R}^n$-valued martingales. Let Y satisfy $|K_s^i|^2 = |K_s^j|^2$ for all j and and $K_s^j\cdot K_s^i = 0$ for $i\neq j$. Suppose $\sqrt{\frac{n+p-2}{p-1}}Y_1$ is differentially subordinate to X. That is,*

$$d\langle \sqrt{\frac{n+p-2}{p-1}}Y_1\rangle_t = \frac{n+p-2}{p-1}|K_t^1|^2 \le \sum_{j=1}^{n}|H_t^j|^2 = d\langle X\rangle_t,$$

a.e. for all $t>0$. Then

$$\|Y\|_p \le (p-1)\|X\|_p, \qquad 2\le p<\infty. \tag{1.7}$$

We need only the $n=2$ case, and this alone is proved. The general case is proved similarly.

The special martingale version for the Beurling-Ahlfors transform is as follows; we refer the reader to [2] for full details. Let $B_t = (Z_t, T-t)$ denote space-time Brownian motion started at $(0,T)$ in $\mathbb{R}^2\times(0,\infty)$. Z_t denotes the usual Brownian motion on the plane.

Let $\varphi\in C_c^\infty(\mathbb{C})$ and $U_\varphi(z,t)$ be the heat extension to the upper half-space. Then

$$U_\varphi(B_t) - U_\varphi(B_0) = \int_0^t \nabla U_\varphi(B_s)\cdot dZ_s$$

is a martingale. Furthermore, if A is a 2×2 matrix, then

$$A\star U_\varphi = \int_0^t A\nabla U_\varphi(B_s)\cdot dZ_s$$

is a martingale transform and hence another martingale.

Denote I as the identity matrix and

$$A = \begin{pmatrix} 1 & i\\ i & -1\end{pmatrix}.$$

Then $B\varphi(Z_T)$ is the conditional expectation or projection of $(A\star U_\varphi)_T$ onto the space of martingales $\{I\star U_f : f\in L^2(\mathbb{C})\}$, under the pseudo-probability measure $dP^T = dP_z dz$. Symbolically,

$$B\varphi(z) = \lim_{T\to\infty} E^T[A \star U_\varphi | B_T = (z,0)].$$

With $Y = A \star U_\varphi$ and $X = \sqrt{\frac{2p}{p-1}} I \star U_\varphi$, we shall verify below that the conditions of Theorem 1.3 are satisfied. Hence

Corollary 1.1. *For* $2 \le p < \infty$*,*

$$\lim_{T\to\infty} \|(A \star U_\varphi)_T\|_p \le \sqrt{2(p^2-p)} \lim_{T\to\infty} \|(I \star U_\varphi)_T\|_p. \tag{1.8}$$

The left hand side is an upper bound for $\|B\varphi\|_p$ since the projection operation is a contraction on L^p. The right hand side limit term is equal to $\|\varphi\|_p$. Again, we refer the reader to [2] for the full details of this argument. This proves Theorem 1.1.

It can be verified by direct computation that when $p = 2$, the constant in (1.8) should be exactly $\sqrt{2}$, hence the above result is not optimal. At the same time, the result gives a clear indication of how much further the martingale direction should take us. In particular, we may *conjecture* that the best constant in (1.8) is $\sqrt{p^2-p}$. The reason for the $\sqrt{2}$ in the constant is that our methods following Burkholder are a pointwise comparison between the quadratic variations of Y and X. This seems to be at present completely fine tuned. However, there may still be some overall integral (expected) cancellation that needs to be explored.

2. Martingale Inequalities

The following theorem is proved in [3]. Recall $p^* = \max\{p, \frac{p}{p-1}\}$.

Theorem 2.1. *Let* X *and* Y *be two* $\mathbb{H}$*-valued continuous-path martingales such that* Y *is differentially subordinate to* X*. Then for* $1 < p < \infty$*,*

$$\|Y\|_p \le (p^*-1)\|X\|_p. \tag{2.1}$$

The constant is best possible.

We quickly review the argument in [3] and point out the modifications needed to obtain Theorem 1.3. Here we assume $\mathbb{H} = \mathbb{R}^2$ and $p \ge 2$. Let

$$V(x,y) = |y|^p - (p^*-1)^p|x|^p.$$

The goal is to prove that $EV(X_t, Y_t) \le 0$ for all $t \ge 0$. As in the work of Burkholder, it is shown instead that $EU(X_t, Y_t) \le 0$ (which proves the same for V) for the alternate function

$$U(x,y) = p(1-1/p^*)^{p-1}(|y| - (p^*-1)|x|)(|x|+|y|)^{p-1}. \tag{2.2}$$

The function U found by Burkholder in [6] has the following structural advantages:

(1) $V(x,y) \le U(x,y)$

(2) For all x, y, h, $k \in \mathbb{H}$, if $|x||y| \neq 0$, then

$$\begin{aligned} &\langle hU_{xx}(x,y), h\rangle + 2\langle hU_{xy}(x,y), k\rangle + \langle kU_{yy}(x,y), k\rangle \\ &\qquad = -c_p(A+B+C), \end{aligned} \tag{2.3}$$

where $c_p > 0$ is a constant depending only on p, and (for $p \geq 2$)

$$A = p(p-1)(|h|^2 - |k|^2)(|x|+|y|)^{p-2}, \tag{2.4}$$

$$B = p(p-2)[|k|^2 - (y', k)^2]|y|^{-1}(|x|+|y|)^{p-1}, \tag{2.5}$$

$$C = p(p-1)(p-2)[(x', h) + (y', k)]^2|x|(|x|+|y|)^{p-3}. \tag{2.6}$$

(3) $U(x,y) \leq 0$ if $|y| < |x|$.

The left side quantity in (2.3) is the directional concavity in direction (h,k). That is, if $G(t) = U(x+ht, y+kt)$, then

$$G''(0) = \langle hU_{xx}(x,y), h\rangle + 2\langle hU_{xy}(x,y), k\rangle + \langle kU_{yy}(x,y), k\rangle.$$

Thus for instance $G''(0) \leq 0$ whenever $|k| \leq |h|$.

Burkholder uses this property to prove the theorem for discrete martingales and later extends to certain continuous cases by approximation. Bañuelos and Wang utilize Itô's formula and make direct use of differential subordination in the following way. Apply Itô's formula to the function U to get

$$\begin{aligned} U(X_t, Y_t) &= U(X_0, Y_0) + \int_0^t \langle U_x(X_s, Y_s), dX_s\rangle \\ &\quad + \int_0^t \langle U_y(X_s, Y_s), dY_s\rangle + \frac{I_t}{2}, \end{aligned}$$

where

$$\begin{aligned} dI = \sum_{i,j=1}^{2} \quad &(U_{x_ix_j}(X,Y)d\langle X_i, X_j\rangle + 2U_{x_iy_j}(X,Y)d\langle X_i, Y_j\rangle \\ &+ U_{y_iy_j}(X,Y)d\langle Y_i, Y_j\rangle). \end{aligned}$$

Recall that we want to show that

$$EU(X_t, Y_t) \leq 0. \tag{2.7}$$

Without loss of generality, assume $|Y_0| < |X_0|$, so that $EU(X_0, Y_0) \leq 0$. Observe next that $\int_0^t \langle U_x(X_s, Y_s), dX_s\rangle$ and $\int_0^t \langle U_y(X_s, Y_s), dY_s\rangle$ are martingales and have expectation 0. Therefore

$$EU(X_t, Y_t) \leq E(I_t/2).$$

Property (2.3) extends to the martingale setting with the replacements $h_ih_j \to d\langle X_i, X_j\rangle$, $k_ik_j \to d\langle Y_i, Y_j\rangle$ and $h_ik_j \to d\langle X_i, Y_j\rangle$. Following Burkholder, observe that the terms B and C are always non-negative and can be left out; it follows from A that

$$I \leq -C_p \int_0^t (|X_s| + |Y_s|)^{p-2} d(\langle X\rangle_s - \langle Y\rangle_s) \leq 0,$$

provided Y is differentially subordinate to X. Hence $EI_t \leq 0$. This proves (2.7) and the Bañuelos–Wang Theorem 2.1 follows.

We now explain the modifications needed to obtain Theorem 1.3. For ease of notation, the time index t is left out for the processes. The hypotheses are: Y satisfies $\langle Y_1 \rangle = \langle Y_2 \rangle$ and $\langle Y_1, Y_2 \rangle = 0$, and $\sqrt{\frac{p}{p-1}} Y_1$ is differentially subordinate to X. As above, our goal is to show $EU(X,Y) \leq 0$. The method adopted in [3] is to drop out B and C, then change the norm square terms to quadratic variation terms. We now include B of (2.5) and verify the calculation again.

The term

$$(y', k)^2 = (k_1 y_1/|y|)^2 + (k_2 y_2/|y|)^2 + 2k_1 k_2 (y_1/|y|)(y_2/|y|)$$

converts in the martingale setting to

$$\left(\frac{Y_1}{|Y|}\right)^2 d\langle Y_1 \rangle + \left(\frac{Y_2}{|Y|}\right)^2 d\langle Y_2 \rangle + 2\frac{Y_1}{|Y|}\frac{Y_2}{|Y|} d\langle Y_1, Y_2 \rangle.$$

Since $d\langle Y_1 \rangle = d\langle Y_2 \rangle$ and $d\langle Y_1, Y_2 \rangle = 0$, this simplifies to $d\langle Y_1 \rangle = \frac{1}{2} d\langle Y \rangle$.

Observe that $|Y|^{-1}(|X| + |Y|)$ is greater than or equal to 1, hence the continuous version of B is bounded below by

$$p(p-2)(|X| + |Y|)^{p-2}(1/2) d\langle Y \rangle.$$

So

$$\begin{aligned} A + B &\geq p(p-1)(|X| + |Y|)^{p-2}\left(d\langle X \rangle - \left(\frac{p}{2(p-1)}\right) d\langle Y \rangle\right) \\ &= p(p-1)(|X| + |Y|)^{p-2}\left(d\langle X \rangle - \left(\frac{p}{p-1}\right) d\langle Y_1 \rangle\right) \\ &\geq 0. \end{aligned}$$

The last inequality is due to differential subordination. It follows that the second order Itô term I_t is non-positive, and $EU(X_t, Y_t) \leq 0$ as required.

3. Subordination of the Beurling–Ahlfors Martingales

We now verify that the martingales $X = \sqrt{\frac{2p}{p-1}} I \star U_\varphi$ and $Y = \int_0^t \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \nabla_z U_\varphi(B_s) \cdot dZ_s$ satisfy the hypotheses of Theorem 1.3. First observe that

$$\begin{aligned} Y \;=\; &\Big(\int_0^t \left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \nabla U_{\varphi_1}(B_s) - \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \nabla U_{\varphi_2}(B_s)\right) \cdot dZ_s, \\ &\int_0^t \left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \nabla U_{\varphi_2}(B_s) + \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \nabla U_{\varphi_1}(B_s)\right) \cdot dZ_s\Big) \end{aligned}$$

From here, it is a straightforward verification that Y satisfies

$$d\langle Y_1 \rangle = d\langle Y_2 \rangle \text{ and } d\langle Y_1, Y_2 \rangle = 0.$$

Recall that the quadratic covariation process of two martingales $\int H_s \cdot dZ_s$ and $\int K_s \cdot dZ_s$ is $\int H_s \cdot K_s ds$. Hence

$$\begin{aligned}\langle Y_1\rangle_t &= \int_0^t \left|\left(\begin{pmatrix}1 & \\ & -1\end{pmatrix}\nabla U_{\varphi_1}(B_s) - \begin{pmatrix} & 1\\ 1 & \end{pmatrix}\nabla U_{\varphi_2}(B_s)\right)\right|^2 ds\\ &= \int_0^t \big((\partial_x U_{\varphi_1}(B_s) - \partial_y U_{\varphi_2}(B_s))^2\\ &\quad + (-\partial_y U_{\varphi_1}(B_s) - \partial_x U_{\varphi_2}(B_s))^2\big) ds\\ &= \int_0^t |\partial U_\varphi|(B_s)^2 ds,\end{aligned}$$

where

$$\begin{aligned}|\partial U_\varphi|^2 &= |(\partial_x + i\partial_y)U_\varphi|^2\\ &= |\nabla U_\varphi|^2 - 2(\partial_x U_{\varphi_1}\partial_y U_{\varphi_2} - \partial_y U_{\varphi_1}\partial_y U_{\varphi_2}).\end{aligned}$$

Next compute the same for

$$\begin{aligned}\langle Y_2\rangle_t &= \int_0^t \left|\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\nabla U_{\varphi_1}(B_s) + \begin{pmatrix}1 & \\ & -1\end{pmatrix}\nabla U_{\varphi_2}(B_s)\right)\right|^2 ds\\ &= \int_0^t \big((\partial_y U_{\varphi_1}(B_s) + \partial_x U_{\varphi_2}(B_s))^2\\ &\quad + (\partial_x U_{\varphi_1}(B_s) - \partial_y U_{\varphi_2}(B_s))^2\big) ds\\ &= \int_0^t |\partial U_\varphi|(B_s)^2 ds,\end{aligned}$$

Hence

$$\langle Y_1\rangle_t = \langle Y_2\rangle_t \quad \text{for all } t \geq 0.$$

Next we compute $\langle Y_1, Y_2\rangle_t$.

$$\begin{aligned}\langle Y_1, Y_2\rangle_t &= \int_0^t \begin{pmatrix}\partial_x U_{\varphi_1} - \partial_y U_{\varphi_2}\\ -\partial_y U_{\varphi_1} - \partial_x U_{\varphi_2}\end{pmatrix} \cdot \begin{pmatrix}\partial_y U_{\varphi_1} + \partial_x U_{\varphi_2}\\ \partial_x U_{\varphi_1} - \partial_y U_{\varphi_2}\end{pmatrix}(B_s) ds\\ &= 0,\end{aligned}$$

as the two interior vectors are orthogonal.

Thus two of the conditions in the hypotheses of Theorem 1.3 are satisfied. The final requirement is that $\sqrt{\frac{p}{p-1}}Y_1$ be differentially subordinate to $X = \sqrt{\frac{2p}{p-1}} I \star U_\varphi$. Since $|\partial U_\varphi|^2 \leq 2|\nabla U_\varphi|^2$, $d\langle Y_1\rangle \leq 2d\langle I \star U_\varphi\rangle$, hence $\sqrt{\frac{p}{p-1}}Y_1$ is indeed differentially subordinate to X.

4. Proof of Theorem 1.2

In this section, the Riesz–Thorin interpolation Theorem is applied using the known L^2 norm: $\|B\|_2 = 1$ and the asymptotic estimate $\|B\|_p \leq \sqrt{2(p^2-p)}$ of Theorem 1.1 to prove Theorem 1.2. We begin with a lemma.

Lemma 4.1.

$$\|B\|_p \leq \begin{cases} \left[\left(\sqrt{2(\tilde{p}^2-\tilde{p})}\right)^{\frac{\tilde{p}}{\tilde{p}-2}}\right]^{\frac{p-2}{p}}; & 2 \leq p < \tilde{p}, \\ \sqrt{2(p^2-p)}; & p \geq \tilde{p}, \end{cases} \tag{4.1}$$

where $\tilde{p}$ *is between* 5.5 *and* 5.6*.*

Proof. Fix $2 < r < \infty$. Then $\|B\|_r \leq \sqrt{2(r^2-r)}$. Also, from the Fourier multiplier formula on $L^2(\mathbb{C})$, $\|B\|_2 = 1$. Therefore the Riesz-Thorin interpolation Theorem [18] gives

$$\|B\|_{r_t} \leq \left(\sqrt{2(r^2-r)}\right)^t, \quad 0 < t < 1, \tag{4.2}$$

where

$$\frac{1}{r_t} = \frac{t}{r} + \frac{1-t}{2}.$$

Solving for t in terms of r_t gives

$$t = \frac{r}{r-2} \cdot \frac{r_t-2}{r_t}.$$

Therefore, (4.2) may be rewritten as

$$\|B\|_{r_t} \leq \left[\left(\sqrt{2(r^2-r)}\right)^{\frac{r}{r-2}}\right]^{\frac{r_t-2}{r_t}}, \quad 0 < t < 1.$$

Since the range of r_t, as $0 < t < 1$, is $(2, r)$ and can be extended as $r_0 = 2$, $r_1 = r$, the above estimate can be stated as

$$\begin{aligned} \|B\|_p &\leq \left[\left(\sqrt{2(r^2-r)}\right)^{\frac{r}{r-2}}\right]^{\frac{p-2}{p}}, \quad 2 \leq p \leq r, \\ &= (\alpha_r)^{\frac{p-2}{p}}. \end{aligned} \tag{4.3}$$

The function $\alpha(r) = \alpha_r = \left(\sqrt{2(r^2-r)}\right)^{\frac{r}{r-2}}$ diverges as $r \to 2$ and as $r \to \infty$ and attains a minimum for some $\tilde{p} \in (2, \infty)$. The minimum can be found by differentiating.

$$\begin{aligned} \alpha'(r) &= \frac{d}{dr}\left[\exp\left\{\frac{r}{r-2}\log\left(\sqrt{2(r^2-r)}\right)\right\}\right] \\ &= \alpha(r)\left[\frac{-2}{(r-2)^2}\log\left(\sqrt{2(r^2-r)}\right) + \frac{r}{r-2}\frac{1}{\sqrt{2(r^2-r)}}\frac{\sqrt{2}(2r-1)}{2\sqrt{r^2-r}}\right] \\ &= \frac{\alpha(r)}{r-2}\left[\frac{-2}{r-2}\log\left(\sqrt{2(r^2-r)}\right) + \frac{2r-1}{2(r-1)}\right]. \end{aligned}$$

It can be deduced using a calculator that $\alpha'(r) = 0$ for some $r := \tilde{p} \in (5.5, 5.6)$. Hence, $\alpha(r)$ is minimized at this $\tilde{p}$, and it follows that the optimal estimate in (4.3) is

$$\|B\|_p \le (\alpha_{\tilde{p}})^{\frac{p-2}{p}}, \quad 2 \le p < \tilde{p}. \tag{4.4}$$

In particular, observe that

$$\alpha_{\tilde{p}}^{\frac{p-2}{p}} \le \alpha_p^{\frac{p-2}{p}} = \sqrt{2(p^2-p)}, \quad 2 \le p < \tilde{p}.$$

This proves the first case in the lemma. For $p \ge \tilde{p}$, the optimal choice for r in (4.3) is $r = p$, so that $\sqrt{2(p^2-p)}$ is the right estimate. This completes the proof of the lemma.

□

Let us now consider the function

$$\beta(p) = \begin{cases} \frac{(\alpha_{\tilde{p}})^{\frac{p-2}{p}}}{p-1}; & 2 \le p < \tilde{p}, \\ \sqrt{\frac{2p}{p-1}}; & p \ge \tilde{p}. \end{cases}$$

Then $\beta(p)$ is an upper bound for $\frac{\|B\|_p}{p-1}$. Since $\beta(p)$ is decreasing in $(\tilde{p}, \infty)$, it suffices to show that the function acquires its maximum value in $[2, \tilde{p}]$ in the interior of the interval. Again calculus gives the results:

$$\begin{aligned} \beta'(p) &= \frac{(p-1)\left[(\alpha_{\tilde{p}})^{\frac{p-2}{p}} \log(\alpha_{\tilde{p}}) \frac{2}{p^2}\right] - (\alpha_{\tilde{p}})^{\frac{p-2}{p}}}{(p-1)^2} \\ &= \alpha_{\tilde{p}}^{\frac{p-2}{p}} \frac{\left[\frac{2(p-1)}{p^2} \log(\alpha_{\tilde{p}}) - 1\right]}{(p-1)^2}. \end{aligned}$$

$\beta'(p) = 0$ when $p^2 - 2\log(\alpha_{\tilde{p}})p + 2\log(\alpha_{\tilde{p}}) = 0$. Using the approximation 5.5 for $\tilde{p}$, the solution for this quadratic equation is

$$\begin{aligned} p' &= \log(\alpha_{\tilde{p}}) + \sqrt{[\log(\alpha_{\tilde{p}})]^2 - 2\log(\alpha_{\tilde{p}})} \\ &\approx 4.87351. \end{aligned}$$

Hence the maximum of $\beta(p)$ is below

$$\beta(4.87351) \approx 1.5738 < 1.575.$$

The latter number is $\frac{63}{40}$ as a fraction. This shows that

$$\|B\|_p \le 1.575\,(p-1), \quad 2 \le p < \infty.$$

The proof of Theorem 1.2 is now finished by duality.

5. Further remarks

The results of Burkholder are ideal in the martingale setting. Several instances of their successes have been pointed out. See further works of Burkholder such as [5], [8], [9], [10], and the book by Stroock [19]. The martingale pair $(X, Y) = (I \star U_\varphi, A \star U_\varphi)$ appears to be different in the sense that pointwise comparison of quadratic variations does not lead to best results. This is not an anomaly; rather we are comparing two martingales of different types: one is actually just the function while the other is a more general extension into the space of martingales. Now consider

$$X_t = \int_0^t \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \nabla_z U_\varphi(B_s) \cdot dZ_s.$$

The projection of this martingale is φ in the same sense as Y projects to $B\varphi$. Moreover, X shares the same properties of Y: $\langle X_1 \rangle = \langle X_2 \rangle = \langle Y_1 \rangle = \langle Y_2 \rangle$ and $\langle X_1, X_2 \rangle = \langle Y_1, Y_2 \rangle = 0$. Let us refer to this, for lack of a better term, as the *2-Brownian property.* The proof of Theorem 1.3 applied to this special case gives

Corollary 5.1. *Let X and Y be two $\mathbb{R}^2$-valued martingales satisfying the 2–Brownian property. Let $p \geq 2$. Then*

$$\sqrt{\frac{2}{p^2 - p}}\|X\|_p \leq \|Y\|_p \leq \sqrt{\frac{p^2 - p}{2}}\|X\|_p. \tag{5.1}$$

Without the improvements of this paper, the constant on the right would be $p - 1$ which is a factor worse for large p. Note that both constants give equality when $p = 2$. It is likely that (5.1) is best possible.

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