# Lévy processes and Fourier multipliers

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#### Abstract

We study Fourier multipliers which result from modulating jumps of Lévy processes. Using the theory of martingale transforms we prove that these operators are bounded in  $L^p(\mathbb{R}^d)$  for 1 and weobtain the same explicit bound for their norm as the one known forthe second order Riesz transforms.

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## 1 Introduction

One of the most basic examples of Calderón–Zygmund singular integrals in  $\mathbb{R}^d$  is the collection of Riesz transforms ([18]),

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \ p.v. \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) \ dy \ , \quad j = 1, 2, \dots, d \ .$$

They can be represented as Fourier multipliers with symbols  $i\xi_j/|\xi|$ ,

$$\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|} \widehat{f}(\xi) \,, \quad f \in L^2(\mathbb{R}^d) \,.$$

Therefore the second order Riesz transforms  $R_j^2$  satisfy

$$\widehat{R_j^2}f(\xi) = -\frac{\xi_j^2}{|\xi|^2}\widehat{f}(\xi) , \quad j = 1, 2, \dots, d.$$
(1)

It follows from the general theory of singular integrals (see Stein [18]) that there exist constants  $C_p$  and  $C'_p$  such that  $||R_jf||_p \leq C_p||f||_p$  and  $||R_j^2f||_p \leq C'_p||f||_p$  for every 1 . There has been considerable interest in recentyears in obtaining the best values for these constants. It was shown in [12] that

$$||R_j f||_p \le \cot\left(\frac{\pi}{2p^*}\right) ||f||_p, \quad f \in L^p(\mathbb{R}^d),$$
(2)

and that  $\cot\left(\frac{\pi}{2p^*}\right)$  is the best (smallest) possible constant for this inequality. Here and below,

$$1 (3)$$

so that

$$p^* - 1 = \max\{p - 1, (p - 1)^{-1}\}.$$

An alternative proof of (2) is given in [3] by applying the martingale transform techniques of Burkholder ([4], [5]) to stochastic integrals obtained from composing harmonic functions with Brownian motion. Using a similar approach, it is also proved in [3] that

$$||R_j R_k f||_p \le (p^* - 1) ||f||_p, \quad j \ne k$$
(4)

and that

$$||R_j^2 f||_p \le (p^* - 1) ||f||_p.$$
(5)

The above operators are closely related to the Beurling–Ahlfors operator

$$Bf(z) = -\frac{1}{\pi} p.v. \int_{\mathcal{C}} \frac{f(\zeta)}{(z-\zeta)^2} d\zeta_1 d\zeta_2.$$
(6)

Indeed, the Beurling–Ahlfors operator is a singular integral of even kernel whose Fourier multiplier is  $\overline{\xi}/\xi$  and hence  $B = -R_1^2 + R_2^2 + 2iR_2R_1$  (see [2]). The computation of the norm of B on  $L^p(\mathbf{C})$  has been a problem of considerable interest for many years now. In [14], Lehto showed that  $\|B\|_p \ge p^* - 1$  and T. Iwaniec conjectured in [11] that  $\|B\|_p = p^* - 1$ .

In [3], the martingale inequalities of Burkholder, together with the representation of the *B* as a conditional expectation of certain stochastic integrals, were used to prove the bound  $||Bf||_p \leq 4(p^*-1)||f||_p$ , for general complex valued *f* and that  $||Bf||_p \leq 2\sqrt{2}(p^*-1)||f||_p$ , for for real valued functions *f*. In [19] Nazarov and Volberg improved the bound to  $2(p^*-1)$  for general *f* and to  $\sqrt{2}(p^*-1)$  for real valued *f* using an analytic (Littlewood–Paley inequalities) approach with Bellman functions that also rests on the martingale inequalities of Burkholder. A different proof of the Nazarov-Volberg bounds was given in [2] using essentially the same proof as the one in [3] but applied to space time Brownian martingales. In [8], Dragičević and Volberg refined the Nazarov-Volberg techniques and obtained that for general *f*,

$$||Bf||_p \le \sqrt{2}(p-1) \left(\frac{1}{2\pi} \int_0^{2\pi} |\cos(\theta)|^p d\theta\right)^{-\frac{1}{p}} ||f||_p, \quad 2 \le p < \infty, \quad (7)$$

and that for real valued f,

$$||Bf||_p \le (p-1) \left(\frac{1}{2\pi} \int_0^{2\pi} |\cos(\theta)|^p d\theta\right)^{-\frac{1}{p}} ||f||_p, \quad 2 \le p < \infty.$$
(8)

By a further refinement of the techniques in [2], it is proved in [1] that

$$||Bf||_p \le \sqrt{2(p^2 - p)} ||f||_p, \quad 2 \le p < \infty,$$
(9)

for general complex valued f, and that

$$||Bf||_p \le \sqrt{p^2 - p} ||f||_p, \quad 2 \le p < \infty,$$
 (10)

for real valued f. Dividing both bounds in (7) and (9) by p and letting this go to infinity both give  $\sqrt{2}$ . However, asymptotically the estimate (9) is slightly better as can be easily checked. Interpolation and the bound in (9) gives the general bound  $||B||_p \leq 1.575(p^*-1)$  for the norm of the operator. For more information on Iwaniec's conjecture and its connections to quasiconformal mappings and other areas of nonlinear PDE, we refer the reader to [12], [11], [3], [19], [8], [2], [1].

The purpose of the present paper is to explore martingale techniques to study Fourier multipliers which arise when the Brownian motion (used to study the Riesz transforms) is replaced by the more general symmetric Lévy process. This leads to a large family of multipliers which generalize the second order Riesz transforms. We obtain the upper bound  $p^* - 1$  for their norms in  $L^p(\mathbb{R}^d)$ , which is the best known to date in the case of the second order Riesz transforms  $R_j^2$ .

Let  $V \ge 0$  be a Lévy measure on  $\mathbb{R}^d$ , that is  $V(\{0\}) = 0, V \ne 0$ , and

$$\int_{\mathbb{R}^d} \min(|x|^2, 1) V(dx) < \infty.$$
(11)

Assume that V is symmetric: V(-B) = V(B). Let  $\phi$  be complex-valued, Borel measurable and symmetric:  $\phi(-z) = \phi(z)$ , and assume that

$$|\phi(z)| \le 1, \quad z \in \mathbb{R}^d.$$

**Theorem 1** The Fourier multiplier with the symbol

$$M(\xi) = \frac{\int_{\mathbb{R}^d} (\cos \xi \cdot z - 1) \phi(z) V(dz)}{\int_{\mathbb{R}^d} (\cos \xi \cdot z - 1) V(dz)},$$
(12)

is bounded on  $L^p(\mathbb{R}^d)$  for  $1 , with the norm at most <math>p^* - 1$ . That is, if we define the operator  $\mathcal{M}$  on  $L^2(\mathbb{R}^d)$  by

$$\widehat{\mathcal{M}f}(\xi) = M(\xi)\widehat{f}(\xi),$$

then  $\mathcal{M}$  has a unique bounded linear extension to  $L^p(\mathbb{R}^d)$ , 1 , and

$$\|\mathcal{M}f\|_{p} \le (p^{*} - 1)\|f\|_{p}.$$
(13)

We note that the boundedness of our multipliers on  $L^p(\mathbb{R}^d)$  does not follow directly from the Hörmander multiplier theorem ([18], page 96) because their symbols (12) generally lack sufficient differentiability. However, for certain special cases (such as those mentioned in (14) below), general  $L^p$  bounds can be obtained from the Marcinkiewicz multiplier theorem, see Stein [18], page 109. It should also be mentioned here that a probabilistic treatment of multipliers under the Hörmander condition is given by McConnell in [15]. While our motivation in this paper is to obtain explicit estimates on the  $L^p$  constants, the motivation for the results in [15] is the desire to extend the classical Hörmander multiplier theorem to functions taking values in Banach spaces with the unconditional martingale difference sequence property, UMD spaces. The technique of [15] use to represent singular integrals and other Fourier multiplies as a conditional expectation of stochastic integrals, has its origins in the paper of Gundy and Varopoulos [10]. This is also the underlying idea in this paper. We however note that all the previous works (including [15], [2] and [1]) rely on the standard Brownian motion or the space-time Brownian motion. To the best of our knowledge, the present paper represents the first application of more general Lévy processes to obtain such results.

A word about our notation. We always assume Borel measurability of considered sets and functions below. By  $L^r = L^r(\mathbb{R}^d)$ , with  $1 \leq r < \infty$ , we will denote the set of complex-valued functions g such that

$$||g||_r = \left[\int_{\mathbb{R}^d} |g(x)|^r dx\right]^{1/r} < \infty,$$

 $L^{\infty}$  are those g for which  $||g||_{\infty} = \sup_{x \in \mathbb{R}^d} |g(x)| < \infty$ , and  $C_c$  consists of continuous compactly supported functions g. For  $g \in L^1$  its Fourier transform is defined as

$$\mathcal{F}g(\xi) = \widehat{g}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot z} g(z) dz , \quad \xi \in \mathbb{R}^d.$$

By Plancherel's Theorem  $\|\widehat{g}\|_2 = (2\pi)^d \|g\|_2$ , and  $\mathcal{F}$  extends to a continuous linear bijection of  $L^2$ . Thus the Fourier multiplier  $\mathcal{M}$  of Theorem 1 has the norm on  $L^2$  equal to  $\|\mathcal{M}\|_{\infty} \leq 1$ , see (12). Theorem 1 will be proved by verifying that (13) holds for every  $f \in C_c$ . This yields that  $\mathcal{M}$  has a unique bounded linear extension to  $L^p$ , denoted also  $\mathcal{M}$ , satisfying (13) for every  $f \in L^p$ .

To give an example, let  $\alpha \in (0, 2)$  and  $j = 1, \ldots, d$ . We have that (13) holds when the multiplier has the symbol

$$M(\xi) = \frac{|\xi_j|^{\alpha}}{|\xi_1|^{\alpha} + \dots + |\xi_d|^{\alpha}}, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$
(14)

We also note that (13) extends to multipliers whose symbols may be obtained as pointwise limits of symbols of the from (12). For instance, (5) can be obtained by letting  $\alpha \to 2$  in (14), see (1). Here is the composition of the paper. The proof of Theorem 1 is given in Section 2. In Section 3 we make some additional consideration, for example we examine (14). The paper is essentially self-contained except for the  $L^p$ estimates for differentially subordinate martingales, which in our case follow form the work of G. Wang [20].

#### 2 Proof of Theorem 1

We first describe the setup which will be used in the proof of the result. Let  $\nu \geq 0$  be a *finite* measure on  $\mathbb{R}^d$  not charging the origin. Assume that  $\nu$  is symmetric:  $\nu(-B) = \nu(B)$ , and  $|\nu| = \nu(\mathbb{R}^d) > 0$ . Let  $\tilde{\nu} = \nu/|\nu|$ . Let **P** and **E** be the probability and expectation for a family of independent random variables  $T_i$  and  $Z_i$ ,  $i = \pm 1, \pm 2, \ldots$ , where each  $T_i$  is exponentially distributed with  $\mathbf{E}T_i = 1/|\nu|$ , and each  $Z_i$  has  $\tilde{\nu}$  as the distribution. We let  $S_i = T_1 + \cdots + T_i$  for  $i = 1, 2, \ldots$ , and  $S_i = -(T_{-1} + \cdots + T_i)$  for  $i = -1, -2, \ldots$  For  $-\infty < s < t < \infty$  we let  $X_{s,t} = \sum_{s < S_i \leq t} Z_i$ , and  $X_{s,t-} = \sum_{s < S_i < t} Z_i$ . We note that  $\mathcal{N}(B) = \#\{i : (S_i, Z_i) \in B\}$  is a Poisson random measure on  $\mathbb{R} \times \mathbb{R}^d$  with intensity measure  $d\nu \nu(dx)$ , and  $X_{s,t} = \int_{s < v \leq t} x \mathcal{N}(dvdx)$  is the Lévy-Itô decomposition of X ([17]). Let  $N(s,t) = \mathcal{N}((s,t] \times \mathbb{R}^d)$  be the number of signals  $S_i$  such that  $s < S_i \leq t$ .

For the reader's convenience we give an elementary proof of what amounts to the Lévy system for X (see [7, VII.68] for more general results).

**Lemma 1** If the Borel measurable function  $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$  is either nonnegative or bounded, and  $s \leq t$ , then

$$\mathbf{E}\sum_{s< S_i \le t} F(S_i, X_{s, S_i}, X_{s, S_i}) = \mathbf{E} \int_s^t \int_{\mathbb{R}^d} F(v, X_{s, v-}, X_{s, v-} + z) \nu(dz) dv.$$
(15)

**PROOF:** Since the arrival time of the *n*-th signal has the gamma distribution,

$$LHS = \sum_{-\infty < i < \infty} \mathbf{E} \left\{ F(S_i, X_{s,S_i-}, X_{s,S_i}) \mathbf{1}_{s < S_i \le t} \right\}$$
$$= \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_s^t F(v, y, y+z) \frac{|\nu|^{n+1} (v-s)^n}{n!} e^{-|\nu|(v-s)} dv \widetilde{\nu}^{*n}(dy) \widetilde{\nu}(dz)$$
$$= \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(v, y, y+z) e^{-|\nu|(v-s)} e^{*(v-s)\nu}(dy) \nu(dz) dv \,.$$

Here  $\mu^{*n}$  is the *n*-fold convolution of a measure  $\mu$  and  $e^{*\mu} = \sum_{n=0}^{\infty} \mu^{*n}/n!$  denotes the convolution exponent of  $\mu$ . In what follows we will use the following two well-known facts.

- 1. First, conditionally on N(s,t) = n, the consecutive signals in (s,t] are uniformly distributed on  $\{(s_1, \ldots, s_n) : s < s_1 \le \ldots \le s_n \le t\}$ .
- 2. Second, let  $s < v \leq t$ . Let  $Tg(v) = \int_{v}^{t} g(u)du$  for measurable and bounded or nonnegative function g. By induction, for  $n = 1, 2, 3, \ldots$ ,

$$T^{n}g(v) = T(T^{n-1}g)(v) = \frac{1}{(n-1)!} \int_{v}^{t} g(u)(u-v)^{n-1} du.$$

We have

$$\begin{aligned} RHS &= \sum_{n=0}^{\infty} \mathbf{E} \left\{ \int_{s}^{t} \int_{\mathbb{R}^{d}} F(v, X_{s,v-}, X_{s,v-} + z) \nu(dz) dv | N(s,t) = n \right\} \frac{|\nu|^{n} (t-s)^{n}}{n!} e^{-|\nu|(t-s)} \\ &= \sum_{n=0}^{\infty} \frac{|\nu|^{n} (t-s)^{n}}{n!} e^{-|\nu|(t-s)} \frac{n!}{(t-s)^{n}} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \int_{s}^{t} ds_{1} \int_{s_{1}}^{t} ds_{2} \dots \int_{s_{n-1}}^{t} ds_{n} \\ &= \sum_{k=0}^{n} \int_{s_{k}}^{s_{k+1}} F(v, y, y+z) dv \tilde{\nu}^{*k}(dy) \nu(dz) \,, \end{aligned}$$

where  $s_0 = s$  and  $s_{k+1} = t$  for k = n. Changing notation involving v and  $s_k$  we obtain

$$\begin{split} &\sum_{n=0}^{\infty} |\nu|^n e^{-|\nu|(t-s)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{k=0}^n \int_s^t ds_1 \dots \int_{s_k}^t ds_{k+1} F(s_{k+1}, y, y+z) \frac{(t-s_{k+1})^{n-k}}{(n-k)!} \tilde{\nu}^{*k}(dy) \nu(dz) \\ &= \sum_{n=0}^{\infty} \frac{|\nu|^n}{n!} e^{-|\nu|(t-s)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_s^t F(v, y, y+z) \sum_{k=0}^n \frac{n!(v-s)^k \tilde{\nu}^{*k}(dy)(t-v)^{n-k}}{k!(n-k)!} \nu(dz) dv \\ &= \sum_{n=0}^{\infty} \frac{|\nu|^n}{n!} e^{-|\nu|(t-s)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_s^t F(v, y, y+z) \left( (v-s) \tilde{\nu} + (t-v) \delta_0 \right)^{*n} (dy) \nu(dz) dv \\ &= \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(v, y, y+z) e^{-|\nu|(t-s)} e^{*((v-s)\tilde{\nu} + (t-v)\delta_0)|\nu|} (dy) \nu(dz) dv \\ &= \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(v, y, y+z) e^{-|\nu|(v-s)} e^{*(v-s)\nu} (dy) \nu(dz) dv = LHS, \end{split}$$

where  $\delta_0$  is the Dirac measure at 0.  $\Box$ 

In particular, for  $s \leq t$  and bounded measurable F we have

$$\mathbf{E} \sum_{s < S_i \le t} \left[ F(S_i, X_{s, S_i}, X_{s, S_i}) - F(S_i, X_{s, S_i}, X_{s, S_i}) \right]$$
  
= 
$$\mathbf{E} \int_s^t \int_{\mathbb{R}^d} \left[ F(v, X_{s, v}, X_{s, v} + z) - F(v, X_{s, v}, X_{s, v}) \right] \nu(dz) dv (16)$$

We will consider the filtration

$$\mathcal{F}_t = \sigma\{X_{s,t}; s \le t\}, \quad t \in \mathbb{R}.$$

For  $t \in \mathbb{R}$  we define

$$p_t = e^{*t(\nu - |\nu|\delta_0)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\nu - |\nu|\delta_0)^{*n} = e^{-t|\nu|} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nu^{*n}.$$
 (17)

The series converges in the norm of absolute variation of measures. Clearly,  $p_t$  is symmetric,

$$\frac{\partial}{\partial t}p_t = (\nu - |\nu|\delta_0) * p_t, \quad t \in \mathbb{R},$$
(18)

and  $p_{t_1} * p_{t_2} = p_{t_1+t_2}$  for  $t_1, t_2 \in \mathbb{R}$ . We have  $p_t \ge 0$  for  $t \ge 0$ , see (17). In fact,  $p_{u-t}$  is the distribution of  $X_{t,u}$ , as well as of  $X_{t,u-}$ , whenever  $t \le u$ . Let

$$\Psi(\xi) = \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1)\nu(dz) \,, \quad \xi \in \mathbb{R}^d \,, \tag{19}$$

where  $\xi \cdot x$  denotes the usual inner product in  $\mathbb{R}^d$ . By symmetry of  $\nu$ ,

$$\Psi(\xi) = \int_{\mathbb{R}^d} (\cos \xi \cdot z - 1) \nu(dz) = \Psi(-\xi) \le 0$$

is real valued for all  $\xi$ . It is also bounded and continuous on  $\mathbb{R}^d$ . We have

$$\hat{p}_t(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(dx) = e^{t\Psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$
(20)

This is the Lévy-Khinchin formula–a direct consequence of (17)–and  $\Psi$  is the corresponding Lévy-Khinchin exponent.

Let  $g \in L^{\infty}$ . For  $x \in \mathbb{R}^d$ ,  $t \leq u$ , we define the *parabolic* extension of g by

$$P_{t,u}g(x) = \int_{\mathbb{R}^d} g(x+y)p_{u-t}(dy) = g*p_{u-t}(x)\,.$$

This equals  $\mathbf{E}g(x + X_{t,u})$ . For  $s \leq t \leq u$  we define the *parabolic martingale* 

$$G_t = G_t(x; s, u; g) = P_{t,u}g(x + X_{s,t}).$$

**Lemma 2**  $G_t$  is a bounded  $\{\mathcal{F}_t\}$ -martingale on  $s \leq t \leq u$ .

**PROOF:** Independence of increments of X yields

$$\mathbf{E}\{g(x+X_{s,u})|\mathcal{F}_t\} = \mathbf{E}\{g(x+X_{s,t}+X_{t,u})|\mathcal{F}_t\} = P_{t,u}g(x+X_{s,t}). \quad \Box$$

Let  $\phi$  be complex-valued and symmetric:  $\phi(-z) = \phi(z)$ , and let  $|\phi| \leq 1$ . For  $x \in \mathbb{R}^d$ ,  $s \leq t \leq u$ , and  $f \in C_c$ , we define  $F_t = F_t(x; s, u; f, \phi)$  as

$$\sum_{s < S_i \le t} \left[ P_{S_i, u} f(x + X_{s, S_i}) - P_{S_i, u} f(x + X_{s, S_i}) \right] \phi(X_{s, S_i} - X_{s, S_i})$$
$$- \int_s^t \int_{\mathbb{R}^d} \left[ P_{v, u} f(x + X_{s, v-} + z) - P_{v, u} f(x + X_{s, v-}) \right] \phi(z) \nu(dz) dv \,.$$

**Lemma 3**  $\mathbf{E}|F_t|^p < \infty$  for very p > 0.

PROOF: Since  $P_{v,u}f$  is bounded for  $v \leq u$ , the continuous (integral) part in the definition of  $F_t$  is bounded. We also see that the jump part (the sum above) is bounded by a constant multiple of N(s,t), which in fact yields exponential integrability of  $F_t$ .  $\Box$ 

In what follows we will denote  $\Delta X_{s,t} = X_{s,t} - X_{s,t-}$ .

**Lemma 4**  $\{F_t\}$  is an  $\{\mathcal{F}_t\}$ -martingale for  $s \leq t \leq u$ .

PROOF: By independence of arrivals of signals  $\{S_i\}$  on disjoint time intervals, and by Lemma 1, for  $s \leq t_1 \leq t_2 \leq u$  we have

$$\begin{split} \mathbf{E} &\left\{ \left[ \sum_{t_1 < S_i \le t_2} \left( P_{S_i, u} f(x + X_{s, S_i}) - P_{S_i, u} f(x + X_{s, S_i -}) \right) \phi(\Delta X_{s, S_i}) \right] | \mathcal{F}_{t_1} \right] \\ &= \mathbf{E} \sum_{t_1 < S_i \le t_2} \left[ P_{S_i, u} f(x' + X_{t_1, S_i}) - P_{S_i, u} f(x' + X_{t_1, S_i -}) \right] \phi(\Delta X_{t_1, S_i}) \\ &= \mathbf{E} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left[ P_{v, u} f(x' + X_{t_1, v -} + z) - P_{v, u} f(x' + X_{t_1, v -}) \right] \phi(z) \nu(dz) dv \,, \end{split}$$

where  $x' = x + X_{s,t_1}$ . This gives the martingale property of F.  $\Box$ 

**Lemma 5**  $G_t(x; s, u; g) = F_t(x; s, u; g, 1) + P_{s,u}g(x).$ 

**PROOF:** Since  $t \mapsto G_t$  is piecewise differentiable with almost surely finite number of discontinuities of the first kind (that is, jumps), we have

$$P_{t,u}g(x+X_{s,t}) - P_{s,u}g(x) = \sum_{s < S_i \le t} [P_{S_i,u}g(x+X_{s,S_i}) - P_{S_i,u}g(x+X_{s,S_i-})] + \int_s^t \frac{\partial}{\partial v} P_{v,u}g(x')dv,$$

where  $x' = x + X_{s,v-}$ . This may be considered a version of the Itô formula ([16]). The proof is concluded by using (18),

$$\frac{\partial}{\partial v} P_{v,u} g(x') = -\int_{\mathbb{R}^d} (\nu - |\nu| \delta_0) (dz) P_{v,u} g(x' + z) \\ = -\int_{\mathbb{R}^d} [P_{v,u} g(x + X_{s,v-} + z) - P_{v,u} g(x + X_{s,v-})] \nu(dz) . \quad \Box$$

Let  $s = t_0 \leq t_1 \leq \ldots \leq t_n = t$ , and  $\sup\{t_i - t_{i-1} : i = 1, \ldots, n\} \to 0$  as  $n \to \infty$ . Since  $F_t$  is square integrable, by orthogonality of increments we have for  $s \leq t \leq u$ ,

$$\mathbf{E}F_t^2 = \mathbf{E}\sum_{i=1}^n (F_{s,t_i} - F_{s,t_{i-1}})^2 \to \mathbf{E}\sum_{s < S_i \le t} [P_{S_i,u}f(x + X_{s,S_i}) - P_{S_i,u}f(x + X_{s,S_{i-1}})]^2 \phi^2(\Delta X_{s,S_i}).$$

The convergence follows from the fact that the integral part of F is Lipschitz continuous. Hence the quadratic variation process of F([7]) is

$$[F,F]_t = \sum_{s < S_i \le t} \left[ P_{S_i,u} f(x + X_{s,S_i}) - P_{S_i,u} f(x + X_{s,S_i-}) \right]^2 \phi^2(\Delta X_{s,S_i}) \,.$$
(21)

By Lemma 5, the quadratic variation of G is

$$[G,G]_t = |P_{s,u}g(x)|^2 + \sum_{s < S_i \le t} \left[ P_{S_i,u}g(x + X_{s,S_i}) - P_{S_i,u}g(x + X_{s,S_i-}) \right]^2.$$
(22)

By (21), polarization, and Lemma 1,

$$\begin{split} \mathbf{E} F_t G_t &= \mathbf{E} F_t(x; s, u; f, \phi) \left[ G_t(x; s, u; g) - P_{s,u}g(x) \right] \\ &= \mathbf{E} \sum_{s < S_i \leq t} \left[ P_{S_i,u} f(x + X_{s,S_i}) - P_{S_i,u} f(x + X_{s,S_i-}) \right] \\ &\quad \left[ P_{S_i,u}g(x + X_{s,S_i}) - P_{S_i,u}g(x + X_{s,S_i-}) \right] \phi(\Delta X_{s,S_i}) \\ &= \mathbf{E} \int_s^t \int_{\mathbb{R}^d} \left[ P_{v,u} f(x + X_{s,v-} + z) - P_{v,u} f(x + X_{s,v-}) \right] \\ &\quad \left[ P_{v,u}g(x + X_{s,v-} + z) - P_{v,u}g(x + X_{s,v-}) \right] \phi(z)\nu(dz)dv \\ &= \int_s^t \int_{\mathbb{R}^d} p_{v-s}(dy) \int_{\mathbb{R}^d} \left[ P_{v,u}g(x + y + z) - P_{v,u}g(x + y) \right] \\ &\quad \left[ P_{v,u}f(x + y + z) - P_{v,u}f(x + y) \right] \phi(z)\nu(dz)dv \,. \end{split}$$

By Fubini's Theorem, for any probability measure  $\mu$  and  $h\in L^1,$ 

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x+y)\mu(dy)dx = \int h(x)dx.$$
 (23)

We define  $|F|_t(x; s, u; f, \phi)$  as

$$\sum_{s < S_i \le t} \left[ P_{S_i, u} |f|(x + X_{s, S_i}) + P_{S_i, u} |f|(x + X_{s, S_i}) \right] |\phi|(\Delta X_{s, S_i}) + \int_s^t \int_{\mathbb{R}^d} \left[ P_{v, u} |f|(x + X_{s, v-} + z) + P_{v, u} |f|(x + X_{s, v-}) \right] |\phi(z)| \nu(dz) dv.$$

By Lemma 1,

$$\mathbf{E}|F|_{t} = 2\mathbf{E}\int_{s}^{t}\int_{\mathbb{R}^{d}} \left[P_{v,u}|f|(x+X_{s,v-}+z) + P_{v,u}|f|(x+X_{s,v-})\right]\nu(dz)dv.$$

Using (23) we obtain

$$\int_{\mathbb{R}^d} \mathbf{E} |F|_t(x; s, u; f, \phi) dx$$

$$= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_s^t \int_{\mathbb{R}^d} \left[ P_{v,u} |f|(x+y+z) + P_{v,u}|f|(x+y) \right] \nu(dz) \, dv \, p_{s,v}(dy) \, dx$$

$$= 4 \int_s^t dv \int_{\mathbb{R}^d} \nu(dz) \int_{\mathbb{R}^d} |f(x)| dx = 4(t-s) |\nu| ||f||_1 < \infty$$
(24)

(compare to Lemma 3). Thus, the following integral is absolutely convergent

$$I_{\phi}(f,g) = \int_{\mathbb{R}^d} \mathbf{E} F_t(x;s,u;f,\phi) G_t(x;s,u;g) dx.$$

We will now consider g = f. By (21), (22) and Lemma 5,  $F_t(x; s, u; f, \phi)$  is differentially subordinate to  $G_t(x; s, u; f)$  in that

$$0 \leq [G,G]_t - [F,F]_t$$
 is non-decreasing for  $t \in [s,u]$ .

Therefore, by [20, Theorem 1], we have that

$$\mathbf{E}|F_t(x;s,u;f,\phi)|^p \le (p^*-1)^p \,\mathbf{E}|G_t(x;s,u;f)|^p \,, \quad s \le t \le u \,.$$
(25)

Here and below we assume (3), in particular 1 .

We note that  $G_u(x; s, u; f) = f(x + X_{s,u})$ . Using (25) and (23) we obtain

$$\int_{\mathbb{R}^d} \mathbf{E} |F_u(x;s,u;f,\phi)|^p dx \le (p^*-1)^p \int_{\mathbb{R}^d} \mathbf{E} |f(x+X_{s,u})|^p dx = (p^*-1)^p ||f||_p^p.$$
(26)

We consider the linear functional

$$L^q \ni g \mapsto \int_{\mathbb{R}^d} \mathbf{E} F_u(x; s, u; f, \phi) g(x + X_{s,u}) dx.$$

By Hölder's inequality, (26) and (23) we have

$$\int_{\mathbb{R}^d} \mathbf{E} |F_u(x; s, u; f, \phi) g(x + X_{s,u})| dx \le (p^* - 1) ||f||_p ||g||_q.$$
(27)

Therefore there is a function  $h \in L^p$  such that

$$\int_{\mathbb{R}^d} \mathbf{E} F_u(x; s, u; f, \phi) g(x + X_{s,u}) dx = \int_{\mathbb{R}^d} h(x) g(x) dx, \quad g \in L^q, \quad (28)$$

and

$$\|h\|_{p} \le (p^{*} - 1)\|f\|_{p}.$$
<sup>(29)</sup>

We also have that  $h \in L^1$ , but the estimate of  $||h||_1$  depends on  $|\nu|$  by (24).

Consider  $\xi \in \mathbb{R}^d$ ,  $e_{\xi}(x) = e^{i\xi \cdot x}$ , and  $\mathcal{E}_t(x; s, u; \xi) = G_t(x; s, u; e_{\xi})$ . To bring about the properties of this martingale we note that by (20)

$$P_{v,u}e_{\xi}(x) = \int_{\mathbb{R}^d} e^{i\xi \cdot (x+y)} p_{u-v}(dy) = e_{\xi}(x)e^{(u-v)\Psi(\xi)}, \quad v \le u.$$

We thus have

$$\mathbf{E}F_{t}\mathcal{E}_{t} = \int_{s}^{t} \int_{\mathbb{R}^{d}} p_{v-s}(dy) \int_{\mathbb{R}^{d}} [P_{v,u}f(x+y+z) - P_{v,u}f(x+y)] e^{(u-v)\Psi(\xi)} e^{i\xi \cdot (x+y)} [e^{i\xi \cdot z} - 1]\phi(z)\nu(dz)dv ,$$

hence

$$I = I_{\phi}(f, e_{\xi}) = \int_{\mathbb{R}^d} \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [P_{v,u}f(x+y+z) - P_{v,u}f(x+y)]$$
$$e^{(u-v)\Psi(\xi)}e^{i\xi \cdot (x+y)}[e^{i\xi \cdot z} - 1]\phi(z)\nu(dz)p_{v-s}(dy)dvdx$$

,

The integral is absolutely convergent by (24). Using (23) and properties of the Fourier transform we obtain

$$\begin{split} I &= \int_{s}^{t} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \left[ P_{v,u} f(x+z) - P_{v,u} f(x) \right] e^{i\xi \cdot x} dx \, e^{(u-v)\Psi(\xi)} [e^{i\xi \cdot z} - 1] \phi(z) \nu(dz) dv \\ &= \int_{s}^{t} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \left[ p_{u-v} * f(x+z) - p_{u-v} * f(x) \right] e^{i\xi \cdot x} dx \, e^{(u-v)\Psi(\xi)} [e^{i\xi \cdot z} - 1] \phi(z) \nu(dz) dv \\ &= \int_{s}^{t} \int_{\mathbb{R}^{d}} \left[ e^{-i\xi \cdot z} e^{(u-v)\Psi(\xi)} \hat{f}(\xi) - e^{(u-v)\Psi(\xi)} \hat{f}(\xi) \right] e^{(u-v)\Psi(\xi)} [e^{i\xi \cdot z} - 1] \phi(z) \nu(dz) dv \\ &= \hat{f}(\xi) \int_{s}^{t} e^{2(u-v)\Psi(\xi)} dv \int_{\mathbb{R}^{d}} |e^{i\xi \cdot z} - 1|^{2} \phi(z) \nu(dz) \, . \end{split}$$

We have  $|e^{i\xi\cdot z}-1|^2 = (\cos\xi\cdot z-1)^2 + \sin^2\xi\cdot z = 2(1-\cos\xi z) = 2\Re(1-e^{i\xi\cdot z})$ . By symmetry of  $\phi\nu$ ,

$$I = \hat{f}(\xi) \left[ e^{2(u-t)\Psi(\xi)} - e^{2(u-s)\Psi(\xi)} \right] \frac{-1}{\Psi(\xi)} \int_{\mathbb{R}^d} (1 - e^{i\xi \cdot z}) \phi(z) \nu(dz) \,, \quad \text{if } \Psi(\xi) < 0 \,,$$

and I = 0 if  $\Psi(\xi) = 0$ . We let t = u = 0, thus obtaining  $I = \hat{f}(\xi)m_s(\xi)$ , where s < 0, and

$$m_s(\xi) = \left[1 - e^{2|s|\Psi(\xi)}\right] \frac{\int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1)\phi(z)\nu(dz)}{\Psi(\xi)}, \quad \text{if } \Psi(\xi) \neq 0, \qquad (30)$$

and  $m_s(\xi) = 0$  if  $\Psi(\xi) = 0$ . From (28) applied to  $g = e_{\xi}$  we obtain

$$\hat{h}(\xi) = m_s(\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$
(31)

Consider the Fourier multiplier  $\mathcal{M}_s$  on  $L^2$  with symbol  $m_s$  (bounded by 1). By (29) the operator uniquely extends to  $L^p$  with norm at most  $p^* - 1$ . Let

$$m(\xi) = \frac{\int (e^{i\xi \cdot z} - 1)\phi(z)\nu(dz)}{\Psi(\xi)}$$
$$= \frac{\int (\cos\xi \cdot z - 1)\phi(z)\nu(dz)}{\int (\cos\xi \cdot z - 1)\nu(dz)}, \quad \text{if } \Psi(\xi) \neq 0$$
(32)

and  $m(\xi) = 0$  if  $\Psi(\xi) = \int (\cos \xi \cdot z - 1)\nu(dz) = 0$ . Clearly,  $m = \lim_{s \to -\infty} m_s$ , pointwise.

Let  $\mathcal{M}$  be the multiplier on  $L^2$  with symbol m. If  $f \in L^2$ , then  $\mathcal{M}_s f \to \mathcal{M}f$  in  $L^2$  by Plancherel's Theorem as  $s \to -\infty$ . By Fatou's Lemma and (29) it follows that  $\|\mathcal{M}f\|_p \leq (p^* - 1)\|f\|_p$ . Therefore  $\mathcal{M}$  extends uniquely from  $C_c$  to  $L^p$  without increasing the norm, which proves Theorem 1 when the Lévy measure is finite.

In the general case let  $\varepsilon > 0$ , and  $\nu(B) = V(B \cap \{|x| > \varepsilon\})$ . For every  $\xi \in \mathbb{R}^d$ , we have that  $\cos \xi \cdot z - 1 \approx -|z|^2/2$  if |z| is small. Using (11) we conclude that  $m(\xi)$  of (32) tends to  $M(\xi)$  of (12) as  $\varepsilon \to 0$ . The latter is *defined* to be zero when its denominator vanishes (see below in this connection). To complete the proof we use the argument as in the preceding paragraph  $\Box$ 

We like to remark that an *antisymmetric*  $\phi$ ,  $\phi(-z) = -\phi(z)$ , yields zero Fourier symbol in Theorem 1 thus our assumption of symmetry of  $\phi$  results in no loss of generality therein. The case of nonsymmetric V, vector-valued  $\phi$ , and space-inhomogeneous V and  $\phi$  require a further development of the method presented in this paper.

### 3 Miscellanea

If  $\Psi(\xi) = \int_{\mathbb{R}^d} (\cos \xi \cdot z - 1) V(dz) = 0$  for  $\xi \neq 0$ , then  $\operatorname{supp} V \subset A_{\xi}$ , where  $A_{\xi} = \{z : \xi \cdot z = 2k\pi \text{ for some integer } k\}.$ 

In particular,  $A_{\xi}$  is discrete in the direction of  $\xi$ . By Fubini's theorem  $\{\xi : \Psi(\xi) = 0\}$  has zero Lebesgue measure. Thus our convention that  $M(\xi) = 0$  when  $\Psi(\xi) = 0$ , does not influence the definition of  $\mathcal{M}$  on  $L^2$  or  $L^p$ . In fact, M does not generally have a limit where  $\Psi(\xi) = 0$ -the behavior of (14) at the origin is rather representative here. Indeed, assume for simplicity of the discussion that V is finite, compactly supported and *nondegenerate*, that is not concentrated on a proper subspace of  $\mathbb{R}^d$ . Let  $\xi \neq 0$  and assume that  $\Psi(\xi) = 0$ . The gradient of  $\Psi(\xi) = \int (e^{i\xi \cdot z} - 1)\phi(z)V(dz)$  is

$$i\int_{\mathbb{R}^d} z e^{i\xi\cdot z} \phi(z) V(dz) = i\int_{A_\xi} z \phi(z) V(dz) = 0$$

and the Jacobian matrix is  $-\int z^T z \,\phi(z) V(dz)$ . Here  $z^T$  denotes the transpose of z. Thus the first nonzero term in the Taylor expansion of  $\Psi(\xi + h)$ 

at  $\xi$  is  $-\frac{1}{2}\int (z \cdot h)^2 V(dz) < 0$  if  $h \neq 0$ . We consider

$$\frac{-\int_{\mathbb{R}^d} (z \cdot h)^2 \phi(z) V(dz)}{-\int_{\mathbb{R}^d} (z \cdot h)^2 V(dz)}$$

The limit of this expression exists if  $h = r\eta$ ,  $\eta \in \mathbb{R}^d \setminus \{0\}$ , and  $r \to 0^+$ , but in general the limit depends on the direction of  $\eta$ , compare (14).

**Example 1** We now examine (14). Let  $\alpha \in (0, 2)$ ,  $j \in \{1, \ldots, d\}$ , and

$$\mu = \delta_{(1,0,\dots,0)} + \delta_{(-1,0,\dots,0)} + \dots + \delta_{(0,0,\dots,1)} + \delta_{(0,0,\dots,-1)}.$$

In polar coordinates we define the Lévy measure  $V(drd\theta) = r^{-1-\alpha}dr\mu(d\theta)$ (of the symmetric  $\alpha$ -stable Lévy process with independent coordinates [17]). We have

$$\Psi(\xi) = c_{\alpha} \int |\xi \cdot z|^{\alpha} \mu(dz)$$

$$= c_{\alpha} \left( |\xi_1|^{\alpha} + \dots + |\xi_d|^{\alpha} \right) ,$$
(33)

where  $c_{\alpha} = -\pi/(2 \sin \frac{\pi \alpha}{2} \Gamma(1+\alpha))$ , see [17, Chapter 14]. Let  $\phi(z_1, \ldots, z_d) = 1$ if  $z_k = 0$  for  $k \neq j$  and  $z_j \neq 0$ , and let  $\phi = 0$  otherwise (we observe only the jumps of the first coordinate process). The symbol (12) becomes (14) with j = 1. By Theorem 1 the corresponding Fourier multiplier has norm bounded by  $p^* - 1$ . Letting  $\alpha \to 2$  we obtain (5) by Fatou's Lemma (see the end of the proof of Theorem 1). Considering  $\phi = a_j$  on the *j*-th coordinate axis (except at the origin) for  $j = 1, \ldots, d$ , we conclude that

$$\|\sum_{j=1}^{d} a_j R_j^2 f\|_p \le (p^* - 1) \|f\|_p, \qquad (34)$$

is valid whenever  $|a_j| \leq 1$ . By considering  $\mu$  concentrated on  $\sqrt{2}/2(\pm 1, \pm 1) \in \mathbb{R}^2$  and suitably chosen  $\phi = \pm 1$  we similarly obtain

$$||2R_j R_k f||_p \le (p^* - 1) ||f||_p, \quad j \ne k.$$
(35)

in dimension d = 2. From this, the upper bound  $2(p^* - 1)$  for the Beurling-Ahlfors operator follows, see Introduction.

**Example 2** Let d = 2 and j = 1 in (14). We have

$$\left|\frac{\partial}{\partial\xi_1}M(\xi)\right|^2 = \alpha^2 \left[\frac{|\xi_2|^{\alpha}}{(|\xi_1|^{\alpha} + |\xi_2|^{\alpha})^2}\right]^2 |\xi_1|^{2(\alpha-1)}.$$

This function is not locally integrable at  $\xi_1 = 0$  if  $0 < \alpha < 1/2$ . Thus the symbol does not satisfy the Hörmander condition ([18]).

Denote  $M(\xi) = \Psi_{\phi}(\xi)/\Psi(\xi)$ , in (12). There is a tempered distribution, say K, with Fourier transform M, such that  $\mathcal{M}\phi = K * \phi$  for smooth compactly supported  $\phi$ . It is of interest to represent  $\mathcal{M}$  as a limit of integrals. Let  $0 < \varepsilon < T < \infty$ . We will approximate M by

$$M_{\varepsilon}^{T}(\xi) = \left[e^{\varepsilon\Psi(\xi)} - e^{T\Psi(\xi)}\right] \frac{\Psi_{\phi}(\xi)}{\Psi(\xi)} = \int_{\varepsilon}^{T} \Psi_{\phi}(\xi) e^{t\Psi(\xi)} dt$$
$$= \int_{\varepsilon}^{T} \Psi_{\phi}(\xi) e^{t\Psi_{\phi}(\xi)} e^{t\Psi_{1-\phi}(\xi)} dt = \int_{\varepsilon}^{T} \left[\frac{d}{dt} e^{t\Psi_{\phi}(\xi)}\right] e^{t\Psi_{1-\phi}(\xi)} dt, \quad (36)$$

where  $\varepsilon \to 0$  and  $T \to \infty$  (compare the proof of Theorem 1). Let  $K_{\varepsilon}^{T}$  be the (tempered) distribution with Fourier transform  $M_{\varepsilon}^{T}$ . If  $0 \le \phi \le 1$ , we consider convolution semigroups  $p_{t}^{\phi}$  and  $p_{t}^{1-\phi}$  of Lévy processes with Levy measures  $\phi V$  and  $(1-\phi)V$ , correspondingly. Motivated by (36) we consider

$$K_{\varepsilon}^{T} = \int_{\varepsilon}^{T} \left[ \frac{d}{dt} p_{t}^{\phi} \right] * p_{t}^{1-\phi} dt \,.$$
(37)

If  $dp_t^{\phi}/dt$  is a finite measure for  $t = \varepsilon$  then it is a finite measure for all  $t \ge \varepsilon$ because  $|dp_t^{\phi}/dt|$  is non-increasing in t. Thus,  $K_{\varepsilon}^T$  is a finite measure and

$$K = \lim_{\varepsilon \to 0, T \to \infty} K_\varepsilon^T \,,$$

as distributions. In passing we like to note that (37) gives an analytic interpretation to our proof of Theorem 1.

**Example 3** When d = 2,  $\alpha = 1$  and j = 1 in (14), the corresponding multiplier is a singular integral

$$\mathcal{M}f(z) = p.v. \int_{\mathbb{R}^2} K(z-w)f(w)dw, \quad z \in \mathbb{R}^2,$$
(38)

understood as above, with the kernel

$$K(x,y) = \frac{-x^2 + y^2 + x^2 \log \left|\frac{x}{y}\right| - y^2 \log \left|\frac{y}{x}\right|}{\pi^2 (x^2 - y^2)^2}, \quad (x,y) \in \mathbb{R}^2.$$
(39)

To obtain (39), we denote

$$p_t(x) = \frac{1}{\pi} \frac{t}{t^2 + x^2}, \quad t > 0, \ x \in \mathbb{R}.$$
 (40)

It is the density function of the one-dimensional symmetric 1-stable Lévy (Cauchy) process on the line. We have  $\hat{p}_t(\xi) = e^{-t|\xi|}$  for  $\xi \in \mathbb{R}$ , and

$$\frac{d}{dt}p_t(x) = \frac{1}{\pi} \frac{-t^2 + x^2}{(t^2 + x^2)^2},$$
(41)

which is integrable for every t > 0. Note that  $p_t(x)p_t(y)$ , for  $(x, y) \in \mathbb{R}^2$ , is the transition density of the Cauchy process with independent coordinates on the plane, compare Example 1. Our discussion above, (40) and (41) yield

$$K(x,y) = \int_0^\infty \frac{t(-t^2 + x^2)}{(t^2 + x^2)^2(t^2 + y^2)} dt$$

Of course, K(x, y) = K(|x|, |y|). By a change of variable,

$$K(hx, hy) = h^{-2}K(x, y)$$
 if  $h > 0$ . (42)

We will determine K(1, y), where y > 1. To this end we observe that

$$\frac{t(-t^2+1)}{(t^2+1)^2(t^2+y^2)} = \frac{2t}{(t^2+1)^2(-1+y^2)} - \frac{t(1+y^2)}{(t^2+1)(-1+y^2)^2} + \frac{t(1+y^2)}{(-1+y^2)^2(t^2+y^2)}$$
  
Integration yields

$$K(1,y) = \frac{-1 + y^2 - (1 + y^2)\log y}{\pi^2(-1 + y^2)^2},$$

and (39) follows by (42).

We note a mild singularity of the kernel K(x, y) at y = 0 in the previous example, in addition to the usual (critical) singularity at (0,0) ([18]). We remark that a stronger singularity may be obtained in higher dimensions within the same setup. The resulting singularities seem amenable by the Calderón-Zygmund theory ([6]), where  $L \log L$  integrability and cancellation of the kernel on the unit sphere are only required to prove the boundedness of  $\mathcal{M}$  on  $L^p$ , 1 . The emphasis in our paper is, however, on obtaininggood estimates of the norm of the operator. Also, (12) goes much beyondhomogeneous symbols ([18]) and gives a wide and natural class of symbolsand singular integrals which deserve a further study. We finally note that $the <math>L^p$  boundedness of our multipliers may have applications to embedding results for anisotropic Sobolev spaces as in [9, Section 2.3], [13, Section 3.1].

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