Lesson 11 on 3.1 Derivatives of polynomials & exponentials

Know $\frac{d}{dx}(x) = 1$, $\frac{d}{dx}(x^2) = 2x$

$f(x) = x^3$ $\quad f'(x) = \lim\limits_{h\to 0} \frac{f(x+h) - f(x)}{h} = 3x^2$

So $f'(x) = 3x^2$

$$DQ = \frac{(x+h)^3 - x^3}{h}$$

Pascal's Δ

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & 3 & & 3 & 1 & & \\ & 1 & 4 & & 6 & & 4 & 1 & \\ & & & & \vdots & & & & \\ 1 & n & & \cdots & & & & n & 1 \end{array}$$

$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \longrightarrow 1 \;\; 4 \;\; 6 \;\; 4 \;\; 1$

nth line $\longrightarrow 1 \;\; n \;\; \cdots \;\; n \;\; 1$

$$= \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h} = 3x^2 + 3xh + h^2$$

$\longrightarrow 3x^2$ as $h \to 0$

$\frac{d}{dx}(x^n)$: $\quad DQ = \frac{(x+h)^n - x^n}{h} = \frac{\left[\cancel{x^n} + nx^{n-1}h + \left(\text{terms with } h^2 \text{ or more}\right)\right] - \cancel{x^n}}{h}$

$= nx^{n-1} + h\cdot(\text{who cares})$

$\longrightarrow nx^{n-1}$ as $h \to 0$.

Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Constant rule: $\frac{d}{dx}\left[c f(x)\right] = c f'(x)$

Sum rule: $\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$

$$DQ_{f+g} = DQ_f + DQ_g$$

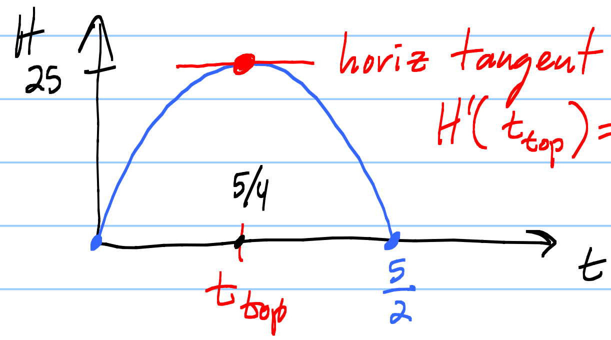
Can differentiate polys!

EX: $\frac{d}{dx}\left(7x^3 + 5x^2 + 3x + 100\right)$

$= 7\cdot(3x^2) + 5\cdot(2x) + 3\cdot(1) + 0$

$= 21x^2 + 10x + 3$

Prob: How high does that ball go? $\rightarrow H(t) = 40t - 16t^2$



$H'(t) = 40\cdot 1 - 16\cdot(2t)$

$0 = 40 - 32t$

$t_{top} = \frac{40}{32} = \frac{5}{4}$ sec

Height at top $= H(t_{top}) = 40\left(\frac{5}{4}\right) - 16\left(\frac{5}{4}\right)^2 = 50 - 25$

$= 25$ ft.

Improved Power Rule:

$$\frac{d}{dx}\left(x^p\right) = p\,x^{p-1}$$

EX: $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = (-1)\,x^{(-1)-1} = -x^{-2} = \frac{-1}{x^2}$

$p = -1$

EX: $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Fact: Power rule works for p rational, even $p < 0$.

Ques: $\frac{d}{dx}\left(x^{\pi}\right) \overset{?}{=} \pi\, x^{\pi - 1}$ Yes!

What is x^{π}? $\quad x^{\pi} = e^{\ln x^{\pi}} = e^{\pi \ln x}$

this is x^{π}

Fact: Power rule works for irrational #s too!

Prob: Find $\dfrac{d}{dx}\left(\underbrace{\dfrac{2\sqrt{x}+\sqrt[3]{x^2}}{x^3}}_{f(x)}\right)$

$$\boxed{\begin{aligned}\sqrt[3]{x^2} &= (x^2)^{1/3}\\ &= x^{2/3}\end{aligned}}$$

$$f(x) = \frac{2x^{1/2}}{x^3} + \frac{x^{2/3}}{x^3}$$

$$= 2x^{\frac{1}{2}-3} + x^{\frac{2}{3}-3}$$

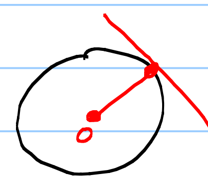
$$f(x) = 2x^{-5/2} + x^{-7/3}$$

So $f'(x) = 2\cdot\left(-\frac{5}{2}\right)x^{-\frac{5}{2}-1} + \left(-\frac{7}{3}\right)x^{-\frac{7}{3}-1}$

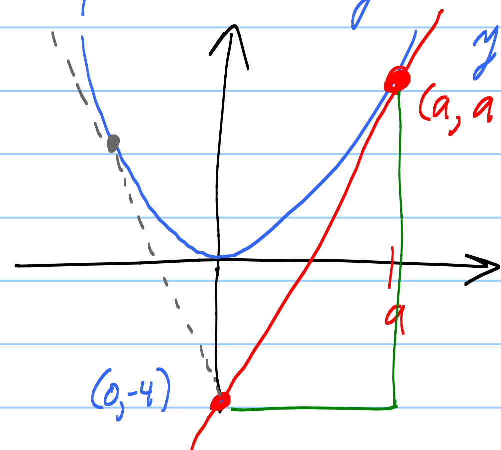
$$= -5x^{-7/2} - \frac{7}{3}x^{-10/3}$$

Note: $x^{-10/3} = \dfrac{1}{x^{10/3}} = \dfrac{1}{\sqrt[3]{x^{10}}}$

Circle



Prob: Find tangent line to parabola $y=x^2$ passing through $(0,-4)$.



$\dfrac{dy}{dx} = 2x$

Tangent line:

Slope = derivative at a

$$\boxed{m = 2a}$$

But m = Slope of line connecting 2 pts $= \dfrac{a^2-(-4)}{a-0} = \dfrac{a^2+4}{a}$

Equate: $2a = \dfrac{a^2+4}{a}$

$2a^2 = a^2+4$

$a^2 = 4$

$$\boxed{a = \pm 2}$$

Find line: $m = \text{slope} = 2a = 2[+2] = 4 \leftarrow \text{slope}$

Point-Slope: $\frac{y-(-4)}{x-0} = 4$ $\qquad (0,-4) \leftarrow \text{point}$

$$y = \underset{\text{slope}}{\underset{\uparrow}{4}}x + \underset{y\text{-intercept}}{\underset{\uparrow}{(-4)}} = mx + b$$

Prob: What is the slope of line $2x+3y+4=0$?

Solve for $y = -\frac{2}{3}x - \frac{4}{3} = \underline{\underline{m}}x + b$ $\qquad$ Slope $= m = -\frac{2}{3}$

Know: $\frac{d}{dx}(e^x) = e^x$

What about $\frac{d}{dx}(2^x)$? $\qquad DQ = \frac{2^{x+h} - 2^x}{h} = \frac{2^x 2^h - 2^x}{h}$

$$DQ = 2^x \underbrace{\frac{2^h - 1}{h}}_{\frac{2^h - 2^0}{h}}$$

$\frac{2^h - 2^0}{h} \longrightarrow$ Slope of tangent line at $x=0$.
Ugly number $= \ln 2$

$$2^x = e^{\ln 2^x} = e^{x \ln 2}$$

What about: $\frac{d}{dx}\left(e^{3x}\right)$

$$DQ = \frac{e^{3(x+h)} - e^{3x}}{h} = \frac{e^{3x+3h} - e^{3x}}{h} = \frac{e^{3x}e^{3h} - e^{3x}}{h}$$

$$= 3e^{3x} \cdot \frac{e^{3h} - 1}{3h}$$

$\frac{e^{\text{small}} - 1}{\text{small}} \rightarrow 1$

as small $\rightarrow$ smaller

Aha! DQ $\rightarrow 3 \cdot e^{3x} \cdot 1$

Formula: $\frac{d}{dx}\left(e^{kx}\right) = k e^{kx}$

$$\frac{d}{dx}\left(2^x\right) = \frac{d}{dx}\left(e^{x \underbrace{\ln 2}_{k}}\right) = \ln 2 \, \underbrace{e^{x \ln 2}}_{2^x}$$

$$= (\ln 2)\, 2^x$$