Lesson 15 on 3.4 Chain rule

EX: $y = \sin(x^4 + x^2) = \sin u$ where $u = x^4 + x^2$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx} = \left(\cos u\right) \cdot \left(4x^3 + 2x\right)$$

$$= \left[\cos(x^4 + x^2)\right] \cdot \left(4x^3 + 2x\right)$$

EX: $y = (x^3 + 5x^2 + x)^{100} = u^{100}$ where $u = x^3 + 5x^2 + x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(100u^{99}\right) \cdot \left(3x^2 + 10x + 1\right)$$

$$= 100\left(x^3 + 5x^2 + x\right)^{99} \cdot \left(3x^2 + 10x + 1\right)$$

Step 1: Diff'tiate the "outside fcn"

Step 2: Multiply by derivative of the "inside fcn"

Step 3: Plug the inner fcn back in.

$$\frac{d}{dx}\left(\text{ℓℓℓℓ}\right)^{100} = 100\left(\text{ℓℓℓℓ}\right)^{99} \frac{d}{dx}\left(\text{ℓℓℓℓ}\right)$$

Can skip naming u fcn with practice.

EX: $y = \cos\left((x^2 + x)^{10}\right) = \cos u$ where $u = (x^2 + x)^{10}$.

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \left(-\sin u\right) \cdot \left[\underbrace{10\left(x^2 + x\right)^9 \cdot (2x + 1)}_{\text{Chain rule, part 2!}}\right]$$

$u = (x^2 + x)^{10}$

$y = \cos u$ where $u = w^{10}$ and $w = x^2 + x$.

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \underbrace{\frac{\cancel{du}}{\cancel{dw}} \frac{\cancel{dw}}{dx}}_{\text{chain rule, part 2.}}$$

Why it works: $h(x) = f(g(x)) \leftarrow f \circ g$

Let $u = g(x)$. $A = g(a)$

f diff'ble: $\dfrac{f(u) - f(A)}{u - A} = f'(A) + E(u)$

where $E(u) \to 0$ as $u \to A$. Define $E(A) = 0$
so $E(u)$ is cont. at A.

$$f(u) - f(A) = f'(A)(u-A) + E(u)(u-A)$$

Substitute $u = g(x)$, $A = g(a)$ in.

$$DQ = \frac{f(g(x)) - f(g(a))}{x - a} = f'(g(a)) \frac{g(x) - g(a)}{x - a} + E(g(x)) \frac{g(x) - g(a)}{x - a}$$

Let $x \to a$ $\longrightarrow f'(g(a)) \cdot g'(a) + 0 \cdot g'(a)$

because g diff'ble at $a \Rightarrow g$ cont. at a,
So $E(g(x)) \to \underbrace{E(g(a))}_{=0}$

$h(x) = f(g(x))$

$$\boxed{h'(x) = f'(g(x))\, g'(x)}$$ Chain rule

$y = f(u)$ where $u = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x))\, g'(x)$$

EX: $y = \dfrac{1}{1+x^2} = (1+x^2)^{-1} = u^{-1}$ where $u = 1+x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[(-1)u^{-1-1}\right] \cdot (0 + 2x)$$

$$= \left(\frac{-1}{u^2}\right) \cdot 2x$$

$$= \frac{-2x}{(1+x^2)^2} \quad \text{(same as Q-rule!)}$$

EX: $y = 2^x = e^{\ln(2^x)} = e^{x \ln 2} = e^u$ where $u = (\ln 2)x$.

$$\frac{dy}{dx} = \left(e^u\right) \cdot \left[(\ln 2)\cdot 1\right] = (\ln 2)\, e^u = (\ln 2)\cdot 2^x$$

$$\left(\text{or } e^u = e^{x\ln 2} = e^{\ln 2^x} = 2^x\right)$$

Fact: $\frac{d}{dx}\left(a^x\right) = (\ln a)\cdot a^x$ (for $a > 0$).

Fact: $\frac{d}{dx}\left(e^{kx}\right) = k e^{kx}$ $\leftarrow$ $\frac{dy}{dx} = ky$

$$\underbrace{\text{Growth rate}} = k \cdot (\text{amount present})$$

$$\begin{pmatrix} k > 0 & \text{Rabbits} \\ k < 0 & \text{Uranium} \end{pmatrix}$$

$y = e^{kx} = e^u$ where $u = kx$.

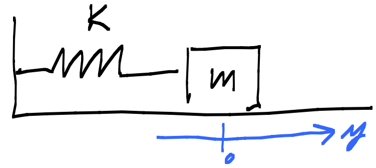
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(e^u\right)\cdot(k\cdot 1) = k e^{kx} \quad \checkmark$$

EX: $y = 10 \sin(3t+1)$

$$\frac{dy}{dt} = 10 \cdot \cos(3t+1)\cdot[3\cdot 1 + 0] = 30 \cos(3t+1)$$

$$\frac{d^2y}{dt^2} = 30\left[-\sin(3t+1)\right](3\cdot 1 + 0) = -90 \sin(3t+1)$$

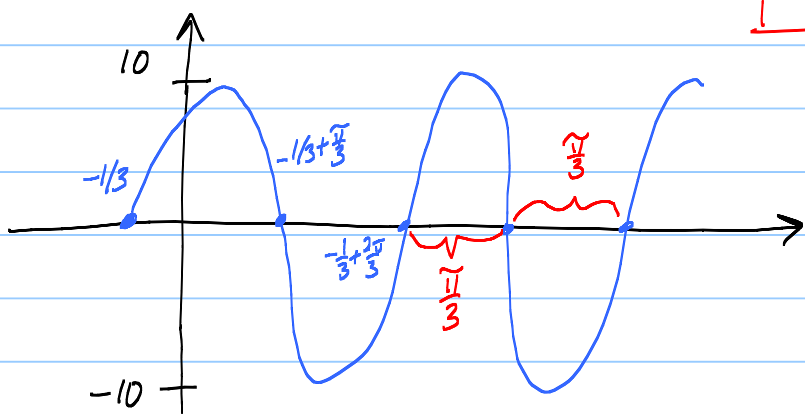
$$\frac{d^2y}{dt^2} = -9\,y$$

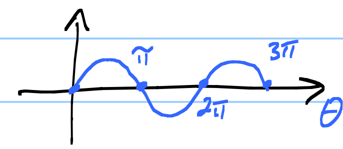


Hooke's Law $\quad F = -Ky$

$$\boxed{m\frac{d^2y}{dt^2} = -Ky}$$

$K = 9$
$m = 1$
$y = 10\sin(3t+1)$





$\sin\theta$

$3t+1 = n\pi$

$t = -\frac{1}{3} + \frac{n\pi}{3}$

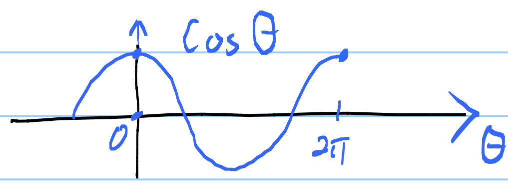
Period $= \frac{2\pi}{3}$

Wiggling 3 times faster than $\sin t$

Shifted $\frac{1}{3}$ to right.

EX: $\quad y = \cos x^2 \leftarrow$ how fast does this wiggle for big x?

$$\frac{dy}{dx} = \left[-\sin(x^2)\right]\cdot 2x = -2x\sin x^2$$



$x^2 = n2\pi \leftarrow$ tops of humps.

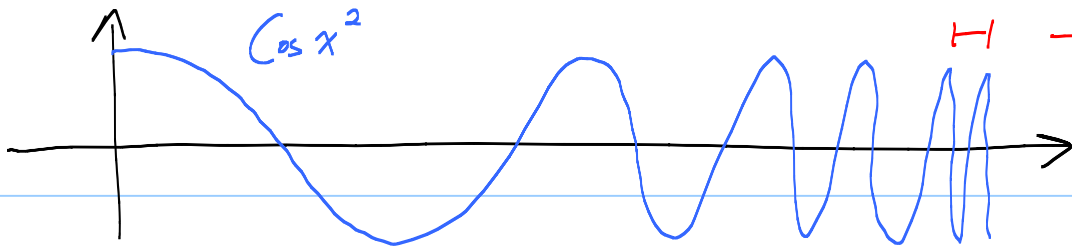
$x = \sqrt{n2\pi}$

distance between hump $n+1$ and hump n:

$$\left(\sqrt{(n+1)2\pi} - \sqrt{n2\pi}\right)\cdot\frac{\sqrt{(n+1)2\pi} + \sqrt{n2\pi}}{\sqrt{(n+1)2\pi} + \sqrt{n2\pi}}$$

$$= \frac{(n+1)2\pi - n2\pi}{\sqrt{} + \sqrt{}} = \frac{2\pi}{\sqrt{(n+1)2\pi} + \sqrt{n2\pi}}$$

$\longrightarrow 0$ as $n \to \infty$!



$\vdash\!\dashv \to 0$ as $n \to \infty$

EX:

$$y = e^{e^{e^x}} = e^u \quad \text{where} \quad u = e^{e^x} = e^w \quad \text{where} \quad w = e^x$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{dy}{dw}\,\frac{dw}{dx} = e^u \cdot e^w \cdot e^x$$

$$= e^{e^{e^x}} \cdot e^{e^x} \cdot e^x$$