Lesson 16 on 3.4 More Chain rule

Exam 1 solutions at
www.math.purdue.edu/~bell/MA161
below the AFTERMATH link

MAPLE

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> diff( tan(exp(2*x)* sin(x^2+1)), x );
      ...., x$10 );
```

Exam 1 : Ave = 76

> 90	A
> 80	B
> 65	C
> 50	D
≤ 50	F

EX: $y = \tan(x^2 e^{3x}) = \tan u$ where $u = \underset{u}{x^2}\,\underset{v}{e^{3x}}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\sec^2 u)\cdot\left((2x)e^{3x} + x^2\cdot e^{3x}\cdot 3\right)$$

$u'v + uv'$

$$= \sec^2(x^2e^{3x})\cdot\left[2xe^{3x} + 3x^2e^{3x}\right]$$

Chain rule:

$h(x) = f(g(x)) = f(u)$ where $u = g(x)$

$h'(x) = f'(g(x))\,g'(x) = f'(u)\,g'(x)$

$= \frac{df}{du}\cdot\frac{du}{dx}$ ← plug $u = g(x)$ back in.

[Range$(g) \subset$ Domain(f)
$g'(a)$ exists and f diff'ble at $g(a)$.

Tables:

$$\frac{d}{dx}\left[u^p\right] = pu^{p-1}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin u\right] = (\cos u)\cdot\frac{du}{dx}$$

Prob:

$g(3) = 7$	$f(3) = 8$	$g'(3) = 100$	$f'(3) = 1000$
$g(7) = 10$	$f(7) = 13$	$g'(7) = 200$	$f'(7) = 2$

$h(x) = f(g(x))$. What is $h'(3)$?

Ans: $h'(3) = f'(\overbrace{g(3)}^{7})\,\overbrace{g'(3)}^{100} = f'(7)\,g'(3) = 2\cdot 100 = 200$

Gut feeling: $\dfrac{f(x)-f(A)}{x-A} = f'(A) + E(x)$ (*)

f diff'ble $\Rightarrow$ $E(x) \to 0$ as $x \to A$.

Solve for f: $f(x) = \underbrace{f(A)+f'(A)(x-A)}_{\text{linear!}} + \underbrace{E(x)(x-A)}_{\text{really small! near } x=A}$

f looks linear under magnification!

Think $f(x) = Mx+B$ $g(x) = mx+b$

$f(g(x)) = M(mx+b)+B = \underbrace{(Mm)}_{}x + (Mb+B)$

Aha! Chain rule!
Slopes multiply.

EX: $e^{\ln x} = x$

$e^{\ln x} = e^u$ where $u = \ln x$

$\dfrac{d}{dx}(e^u) = \dfrac{d}{dx}(x) = 1$

$= e^u \dfrac{du}{dx} = 1$

$\underbrace{e^{\ln x}}_{x} \dfrac{d}{dx}(\ln x) = 1$ Aha! $\boxed{\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}}$

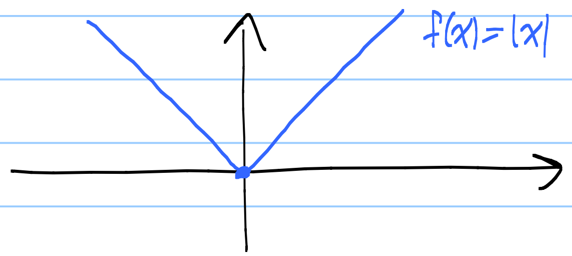
Important fact because the power rule $\dfrac{d}{dx}(x^p) = px^{p-1}$

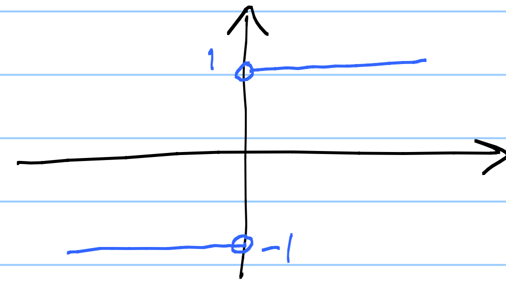
get all powers except $=-1$.

$p=0$ case: $\dfrac{d}{dx}(\underbrace{x^0}_{1}) = 0$

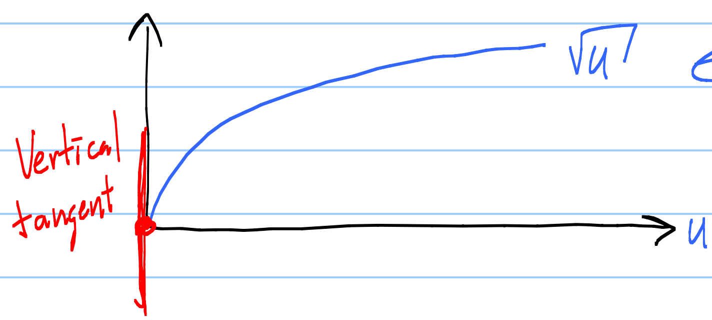
EX: $y = |x| = \sqrt{x^2} = \underset{= u^{1/2}}{\sqrt{u}}$ where $u = x^2$.

$$\frac{dy}{dx} = \frac{1}{2}(u)^{\frac{1}{2}-1} \cdot (2x) = \frac{2x}{2\sqrt{x^2}} = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$





Chain rule. What goes wrong at $x=0$?



$\leftarrow$ not diff'ble at $u=0$!

$$\frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}}$$

EX: $y = \cos^2 x + \cos x^2 = (\cos x)^2 + \cos(x^2)$

$$\frac{dy}{dx} = 2(\cos x)^1 \cdot (-\sin x) + [-\sin(x^2)] \cdot (2x)$$

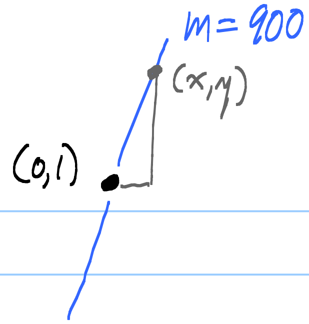
$$= -2\sin x \cos x - 2x \sin x^2$$

Prob: Find eqn of tangent line to $y = (1+9x)^{100}$ above $x=0$.

Point is $x=0$: $y = (1+9\cdot 0)^{100} = 1^{100} = 1$.

What tangent line at $(0,1)$.

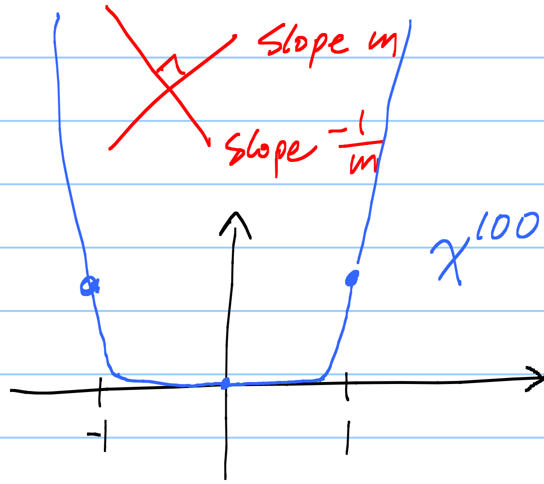
Slope $= \frac{dy}{dx}$ at $x=0$ $= 100(1+9x)^{99} \cdot (0+9) \Big|_{x=0}$



$$= 100(1+9\cdot 0)^{99}\cdot 9 = 900$$

$$\frac{y-1}{x-0} = 900 \qquad \boxed{y = 900x + 1}$$

Perpendicular line to tangent is called the normal. Slope $= \frac{-1}{900}$



x^{100} looks square!

EX: $y = \sin(\sin x) = \sin(\sin x)$

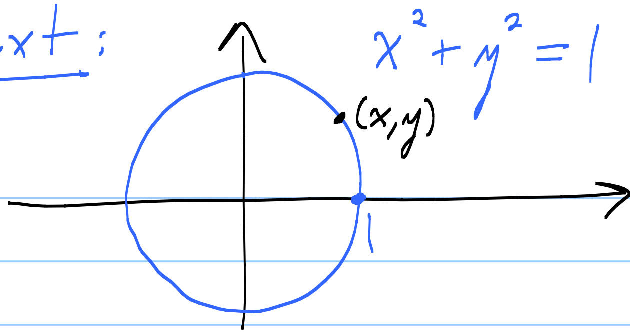
$$\frac{dy}{dx} = \cos(\sin x)\cdot(\cos x)$$

EX: $y = \dfrac{1}{\sqrt[5]{1+\csc x}} = (1+\csc x)^{-1/5}$

$$\frac{dy}{dx} = -\frac{1}{5}(1+\csc x)^{-\frac{1}{5}-1}(0 - \csc x \cot x)$$

$$= \frac{\csc x \cot x}{5(1+\csc x)^{6/5}}$$

What next:



$x^2+y^2=1$ ← implicit definition of $y=f(x)$

$y^2=1-x^2$

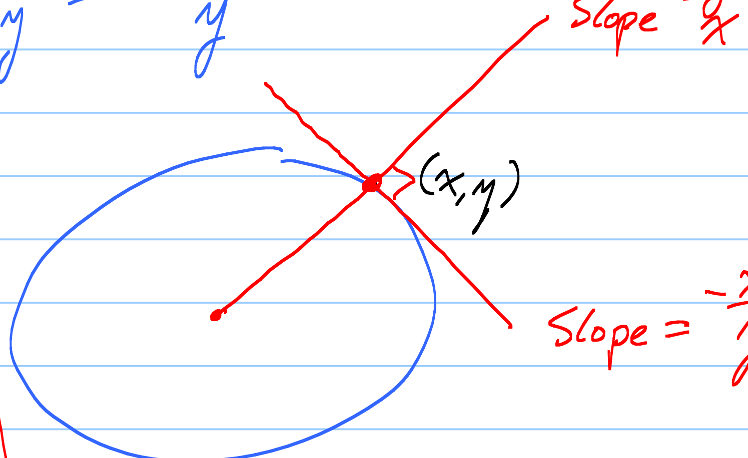
$y=\pm\sqrt{1-x^2}$

Hmmm. $$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[1]=0$$

$$2x+\underbrace{2y\frac{dy}{dx}}_{\text{Chain Rule}}=0$$

$$\frac{d}{dx}\left[u^p\right]=pu^{p-1}\frac{du}{dx}$$

Solve for $$\frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y}$$



$$y=\sqrt{1-x^2}=(1-x^2)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}(1-x^2)^{\frac{1}{2}-1}\cdot(0-2x)$$

$$=\frac{-x}{\sqrt{1-x^2}}$$

$y \rightarrow \sqrt{1-x^2}$