Lesson 17 on 3.5 Implicit differentiation

$$\frac{d}{dx}(eee)^n = n(eee)^{n-1}\frac{d}{dx}(eee)$$

$$\frac{d}{dx}(\sin x)^n = n(\sin x)^{n-1}[\cos x]$$

$$\frac{d}{dx}(y^n) = n y^{n-1}\frac{dy}{dx}$$

Ex: $\frac{d}{dx}(\underset{u}{x^2}\,\underset{v}{y^4}) = u'v + uv' = (2x)y^4 + (x^2)\left[4y^3\frac{dy}{dx}\right]$

Ex: $x^4 + y^4 = 1$ ← defines a function $y = f(x)$ of x implicitly.

$$\frac{d}{dx}\left[x^4 + y^4\right] = \frac{d}{dx}[1]$$

$$4x^3 + 4y^3\cdot\frac{dy}{dx} = 0$$ ← solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$$

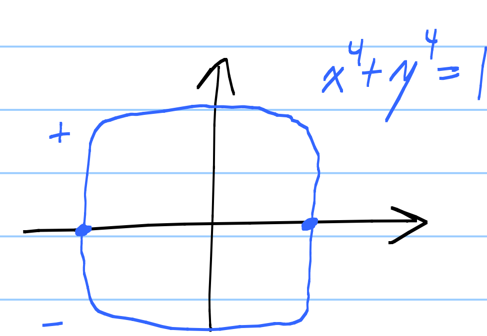
$$\boxed{\frac{dy}{dx} = \frac{-x^3}{y^3}}$$

Check: $x^4 + y^4 = 1$

$y^4 = 1 - x^4$

$y = \pm\sqrt[4]{1-x^4}$ ← explicit y

$y = (1-x^4)^{1/4}$ (top half.)



$$\frac{dy}{dx} = \frac{1}{4}(1-x^4)^{-3/4}\left[0-4x^3\right] = \frac{-x^3}{(1-x^4)^{3/4}} = \frac{-x^3}{y^3}$$ ✓

EX: $\sin(x+y) = 2x - 2y$ ← defines y as a fcn of x implicitly.

$$\frac{d}{dx}\left[\sin(x+y)\right] = \frac{d}{dx}(2x-2y)$$

$$\cos(x+y)\cdot\left[1+\frac{dy}{dx}\right] = 2 - 2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[\cos(x+y) + 2\right] = 2 - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{2-\cos(x+y)}{2+\cos(x+y)}}$$

$$\left.\frac{dy}{dx}\right|_{(\pi,\pi)} = \frac{2-\cos(\pi+\pi)}{2+\cos(\pi+\pi)} = \frac{2-1}{2+1} = \underline{\underline{\frac{1}{3}}}$$

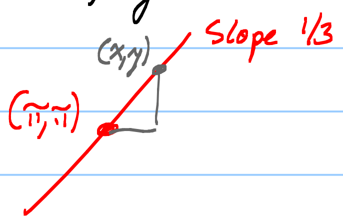
Prob: (π,π) is on the graph;

$$\sin(x+y) = 2x-2y$$

$$\sin(\underbrace{\pi+\pi}_{2\pi}) \stackrel{?}{=} 2\pi - 2\pi$$

$$0 \stackrel{\checkmark}{=} 0$$

Find tangent line to $y=f(x)$ at $x=\pi$, $y=\pi$.



$$\frac{y-\pi}{x-\pi} = 1/3$$

$$y = \pi + \frac{1}{3}(x-\pi)$$

$$\underbrace{f(a) + f'(a)(x-a)}_{\text{linear approx to } f \text{ near } a}$$

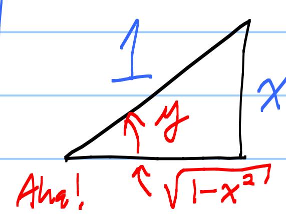
Derivatives of inverse trig fcns

1) $y = \sin^{-1} x$ ← explicit formula for y

$x = \sin y$ ← implicit for y.

$$\frac{d}{dx}[x] = \frac{d}{dx}(\sin y) = (\cos y)\cdot\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos y}} \underset{\text{Aha!}}{=} \frac{1}{\left(\frac{\sqrt{1-x^2}}{1}\right)} = \frac{1}{\sqrt{1-x^2}}$$



$\sin y = x$

$$\boxed{\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}} \leftarrow \text{memorize}$$

Book: $\cos^2 y + \sin^2 y = 1$

$\cos y = \sqrt{1 - \underbrace{\sin^2 y}_{x^2}}$ But $x = \sin y$.

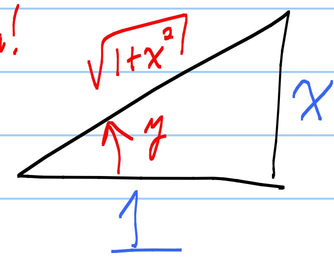
$\cos y = \sqrt{1-x^2}$ ✓

2) $y = \tan^{-1} x$

$x = \tan y$

$1 = \left(\sec^2 y\right) \cdot \frac{dy}{dx}$

Aha!

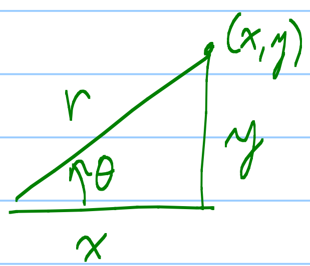


$\tan y = \frac{x}{1} = x$ ✓

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \underset{\substack{\uparrow \\ \text{Aha!}}}{=} \frac{1}{\left(\frac{\sqrt{1+x^2}}{1}\right)^2} = \frac{1}{1+x^2}$$

$$\boxed{\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}} \leftarrow \text{memorize}$$

Trig:



Pyth. Thm. $x^2 + y^2 = r^2$

1) Divide by r^2 :

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

2) Divide by x^2: $\left(\frac{x}{x}\right)^2 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

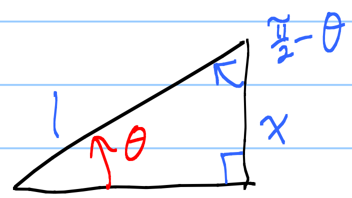
3) Divide by y^2: $\left(\frac{x}{y}\right)^2 + \left(\frac{y}{y}\right)^2 = \left(\frac{r}{y}\right)^2$

$$\boxed{\cot^2\theta + 1 = \csc^2\theta}$$

$x = \tan y$: $\quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$

$$= \frac{1}{1+x^2} \checkmark$$

Why almost nobody talks about $\cos^{-1}x$:



$$\theta = \sin^{-1}x$$

$$\frac{\pi}{2} - \theta = \cos^{-1}x \quad +$$

$$\frac{\pi}{2} = \sin^{-1}x + \cos^{-1}x$$

Aha! $\quad \frac{d}{dx}\left(\cos^{-1}x\right) = \frac{d}{dx}\left[\frac{\pi}{2} - \sin^{-1}x\right] = 0 - \frac{1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}}$$

WebAssign obstacles: $\quad \frac{d}{dx} \quad \ldots + e^{x^2y^3} \ldots \quad = 0$

$$\frac{d}{dx}\left(e^{x^2y^3}\right) = \frac{d}{dx}e^u \quad \text{where } u = x^2y^3$$

$$= e^u \cdot \frac{du}{dx} = e^{x^2 y^3} \cdot \left[(2x) y^3 + x^2 \cdot 3y^2 \frac{dy}{dx} \right]$$

$$\frac{d}{dx}\left(e^{x/y}\right) = e^{x/y} \cdot \frac{d}{dx}\left(\frac{x}{y}\right) = e^{xy} \underbrace{\left[\frac{(1)\cdot y - x \frac{dy}{dx}}{y^2} \right]}_{\text{Quotient rule.}}$$

u, v

Can we get y'' by I.D.?

$$x^2 + y^2 = 1$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Aha!

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x}{y}\right) = \frac{(-1)y - (-x)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y + x\left[\frac{-x}{y}\right]}{y^2} = -\frac{x^2+y^2}{y^3} = \frac{-1}{y^3}$$

$x^2 + y^2 = 1$

Check: $y = \sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{(1-x^2)^{3/2}} \checkmark$$