Lesson 18 on 3.6 Derivatives of logarithmic functions

$y = \ln x$ ← explicit

$x = e^y$ ← implicit

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}\left(e^y\right) = e^y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}}$$ for $x > 0$!

$$\boxed{\frac{d}{dx}\left(\ln u\right) = \frac{1}{u} \cdot \frac{du}{dx}}$$

What if $x < 0$? Hmmm.

$$y = \ln(-x)$$

$$\frac{dy}{dx} = \frac{1}{(-x)} \cdot \underbrace{\frac{d}{dx}(-x)}_{-1}$$

$$= \frac{1}{x}$$

$x < 0$: $\ln(-x) = \ln|x|$

$x > 0$: $\ln x = \ln|x|$

Famous formula: $$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad (x \neq 0)$$

$$\rightarrow \boxed{\frac{d}{dx} \ln|u| = \frac{1}{u} \cdot \frac{du}{dx}}$$

$$\frac{d}{dx}\left(\ln|\underbrace{\sin x}_{u}|\right) = \frac{1}{\sin x} \cdot (\cos x) = \frac{\cos x}{\sin x} = \cot x$$

$$\frac{d}{dx}\left(\ln|\cos x|\right) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x \qquad (*)$$

Prob: Find $f(x)$ with $f'(x) = \tan x$.

Just take minus of $(*)$!

$$\frac{d}{dx}\Big(\underbrace{\underbrace{-\ln|\cos x|}_{\ln|\cos x|^{-1}}}_{\ln|\sec x|}\Big) = \tan x$$

$$\boxed{\frac{d}{dx}\big(\ln|\sec x|\big) = \tan x}$$

Hmmm. $y = 100 + \ln|\sec x|$ works too.

Fact: $\ln|\sec x| + C$ where C is any constant are all the possible solutions.

Logarithmic differentiation: $y = \dfrac{x^2 e^{2x} \sin x}{x^2+1}$

Step 1: Take $\ln$:

$$\ln y = \underbrace{\ln x^2}_{2\ln x} + \underbrace{\ln e^{2x}}_{2x} + \ln \sin x - \ln\big(\underbrace{x^2+1}_{u}\big)$$

$$\boxed{\begin{aligned} \ln\frac{abc}{d} &= \ln(abc) - \ln d \\ &= \ln a + \ln b + \ln c - \ln d \end{aligned}}$$

Step 2: Diff wrt x:

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{1}{x} + 2 + \cot x - \frac{1}{(x^2+1)}\cdot(2x)$$

Step 3: Solve for $\frac{dy}{dx}$: $\quad \dfrac{dy}{dx} = y\cdot\big(\ \text{eeee}\ \big)$

Step 4: Plug y back in: $\quad \dfrac{dy}{dx} = \dfrac{x^2e^{2x}\sin x}{x^2+1}\left(\dfrac{2}{x} + 2 + \cot x - \dfrac{2x}{x^2+1}\right)$

EX: $y = x^x$

$$\boxed{\ln a^b = b\ln a}$$

1. $\ln y = \ln x^x = \overset{u}{x}\,\overset{v}{\ln x}$

2. $\dfrac{1}{y}\cdot\dfrac{dy}{dx} = u'v + uv' = (1)\ln x + x\cdot\left(\dfrac{1}{x}\right)$

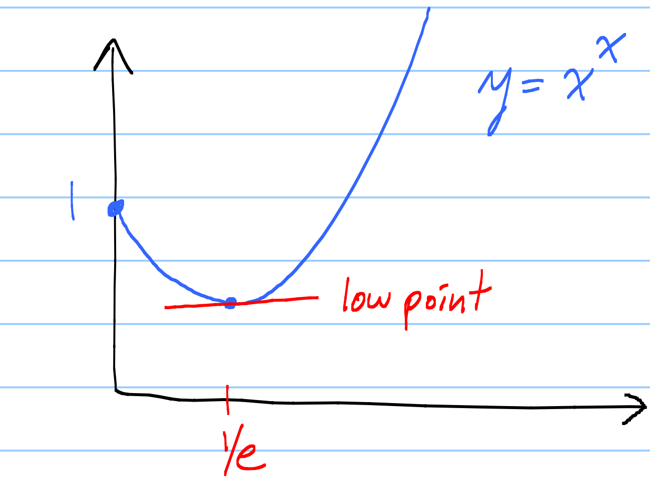
$$= (\ln x) + 1$$

3. $\dfrac{dy}{dx} = y \cdot (\text{ee}) = x^x (1 + \ln x)$

$$\boxed{\frac{d}{dx}\left(x^x\right) = x^x (1 + \ln x)}$$

$$\begin{cases} \dfrac{d}{dx}\left(a^x\right) = (\ln a)\, a^x & (a > 0,\ a \text{ const.}) \\[2ex] \dfrac{d}{dx}\left(x^a\right) = a x^{a-1} \end{cases}$$

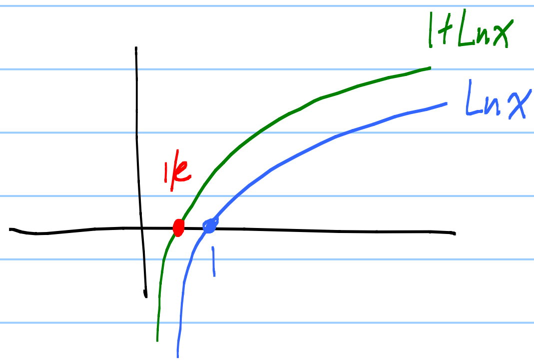
Let $a = x$, Add formulas, Get

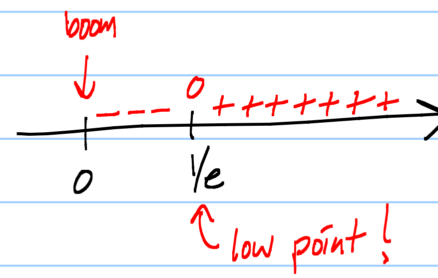


$$\frac{dy}{dx} = \underset{\substack{\uparrow \\ \text{never zero}}}{x^x} \Big(\underbrace{1 + \ln x}_{=0}\Big) = 0$$

$$\underbrace{e^{\ln x}}_{x} = e^{-1}$$

$$x = e^{-1} = 1/e$$





L'Hôpital's Rule will show $\displaystyle\lim_{x \to 0+} x^x = 1$

0^0 is indeterminant form.

$$x^x = e^{\ln x^x} = e^{x \ln x}$$

$x \ln x \quad 0\cdot(-\infty)$

$\to 0$ as $x \to 0+$ by L'H.

So $e^{x\ln x} \to 1$ as $x \to 0+$

Mega product rule: $\quad y = f_1(x)\, f_2(x) \cdots f_N(x)$

$$\ln y = \ln f_1 + \ln f_2 + \cdots + \ln f_N$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_N'}{f_N}$$

$$\frac{dy}{dx} = y\cdot(\text{Sum}) = (f_1 f_2 \cdots f_N)\left(\frac{f_1'}{f_1} + \cdots + \frac{f_N'}{f_N}\right)$$

$$= \underset{k=1}{\overset{N}{\text{Sum}}}\; f_k' \,(\text{Product of } f_j\text{'s},\ j \neq k)$$

EX: $\quad y = \sqrt[3]{\dfrac{1-x^2}{1+x^2}} = \left(\dfrac{1-x^2}{1+x^2}\right)^{1/3}$

$\ln a^b = b \ln a$

$\ln \frac{a}{b} = \ln a - \ln b$

$$\ln y = \frac{1}{3}\ln\left(\frac{1-x^2}{1+x^2}\right) = \frac{1}{3}\ln(\underbrace{1-x^2}_{u}) - \frac{1}{3}\ln(1+x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\cdot\frac{1}{1-x^2}\cdot(-2x) - \frac{1}{3}\cdot\frac{1}{1+x^2}\cdot(2x)$$

$$= -\frac{2}{3}\Big[\underbrace{\frac{x}{1-x^2} + \frac{x}{1+x^2}}_{\frac{2x}{1-x^4}}\Big] = -\frac{4}{3}\,\frac{x}{1-x^4}$$

$$\frac{dy}{dx} = y\cdot(\text{eee}) = \sqrt[3]{\frac{1-x^2}{1+x^2}}\cdot\left(-\frac{4}{3}\,\frac{x}{1-x^4}\right)$$

WebAssign: $y = (x^4+1)^5 (x^2+x+1)^{10}$ ← easier to use product rule, power rule.

$$\left(1+\frac{1}{n}\right)^n \longrightarrow e \quad \text{as} \quad n \to \infty.$$