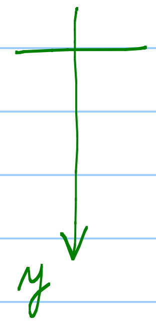
Lesson 19 on 3.7 & 3.11 Hyperbolic functions

$$\frac{dy}{dx} = \begin{cases} \text{slope of tangent} \\ \text{or} \\ \text{instantaneous velocity} \quad (y = \text{position},\ x = \text{time}) \\ \text{or} \\ \text{rate of change} \end{cases}$$

$F = ma$ Force due to gravity near the surface of the earth feels constant. Acceleration: $a = \frac{1}{m} F$ is constant too.

Prob: Drop a rock:



$0 \leftarrow$ top of math.

$y = f(t)$

$v = \frac{dy}{dt} = f'(t)$

$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = f''(t)$

$a = 32\ (\text{ft/sec})/\text{sec}$

$$\boxed{a = 32\ \text{ft}/\text{sec}^2}$$

Want velocity v to satisfy $\frac{dv}{dt} = a = 32$

Hmmm. $v = 32t$ works. So does $\underline{v = 32t + C}$, C const.

But also need $v = 0$ when $t = 0$.

$32 \cdot 0 + C = 0 \leftarrow$ Gives $C = 0$.

$\boxed{v = 32t}$ works.

Fact: Solution to $\begin{cases} \frac{dv}{dt} = 32 \\ v = 0 \text{ at } t = 0 \end{cases}$ is unique!

So $v = 32t$ is it!

Next: Want y with $\frac{dy}{dt} = v = 32t$.

Hmmm. $\frac{d}{dt}(ct^2) = \underbrace{2c}_{32}\, t$ $\quad c = 16$.

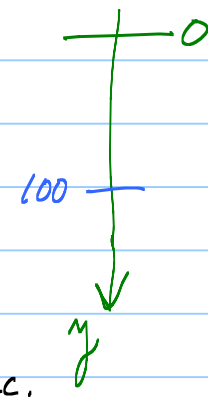
$y = 16t^2$ does it. So does $y = 16t^2 + C$.

But I also need $y = 0$ when $t = 0$.

$16 \cdot 0^2 + C = 0 \leftarrow$ Gives $C = 0$.

$\boxed{y = 16t^2} \leftarrow$ sol^n!

Prob: Drop rock from 100 ft. How fast is it going when it hits ground?



$a = 32$
$v = 32t$
$y = 16t^2$

Question 1: When does it hit? When $16t^2 = 100$

$t = \sqrt{\frac{100}{16}} = \frac{10}{4}$
$= \frac{5}{2} = 2.5$ sec.

Question 2: What is v when $t = \frac{5}{2}$?

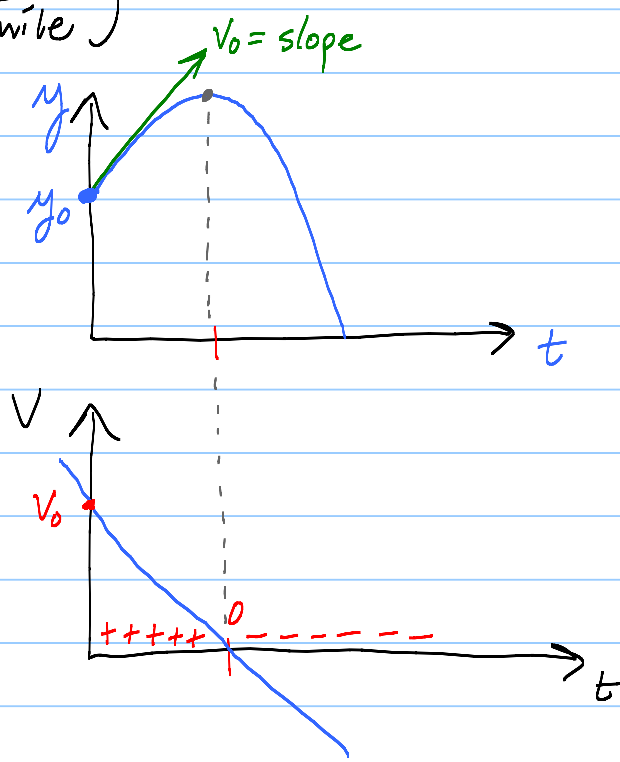
$v = 32t = 32 \cdot \frac{5}{2} = 80$ ft/sec

$= 80 \frac{\text{ft}}{\text{sec}} \frac{60 \cdot 60 \text{ sec/hr}}{5280 \text{ ft/mile}}$

$= 55$ mph.

$$\frac{\left(\frac{\text{ft}}{\text{sec}} \cdot \frac{\text{sec}}{\text{hr}}\right)}{\left(\frac{\text{ft}}{\text{mile}}\right)} = \frac{\text{miles}}{\text{hr}}$$

EX:



$a = -32$
$v = -32t + v_0$
$y = -16t^2 + v_0 t + y_0$

Hyperbolic fcns: First Trig fcns:

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \quad +$$
$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Hyperbolic versions :

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$
$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}$$

Formulas :

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\frac{1}{2}e^{x} - \frac{1}{2}e^{\overset{u}{-x}}\right] = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}(-1)$$
$$= \frac{1}{2}(e^{x} + e^{-x})$$
$$= \cosh x \quad !$$

But

$$\frac{d}{dx}(\cosh x) = \sinh x$$

no − here!

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\boxed{\cosh^2 x - \sinh^2 x \equiv 1}$$

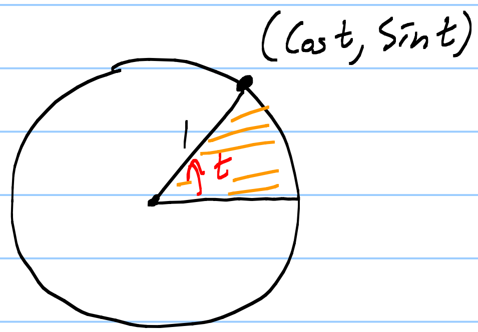
$$\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2 \overset{?}{=} 1$$

$e^{-x} = 1/e^{x}$

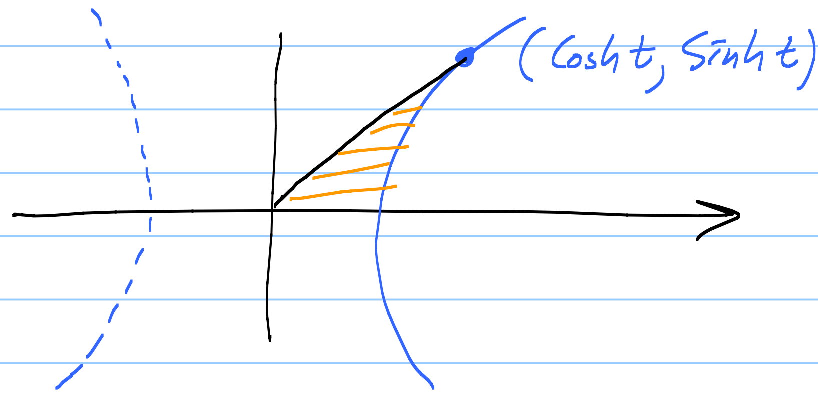
$$\frac{1}{4}\left[\underbrace{e^{x} \cdot e^{x}}_{e^{2x}} + \underbrace{2e^{x}e^{-x}}_{2\cdot 1} + \underbrace{e^{-x}e^{-x}}_{e^{-2x}}\right] - \frac{1}{4}\left[e^{2x} - 2 + e^{-2x}\right] \overset{?}{=} 1 \checkmark$$

$$\frac{d}{dx}(\operatorname{Tanh} x) = \frac{d}{dx}\left(\frac{\operatorname{Sinh} x}{\operatorname{Cosh} x}\right) = \frac{\overbrace{(\operatorname{Cosh} x)(\operatorname{Cosh} x) - (\operatorname{Sinh} x)(\operatorname{Sinh} x)}^{1}}{\operatorname{Cosh}^2 x}$$

$$= \frac{1}{\operatorname{Cosh}^2 x} \quad \checkmark$$



$x^2 + y^2 = 1$

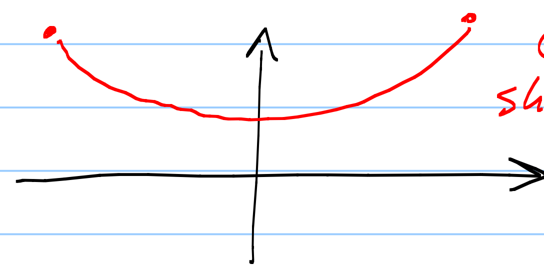


$x^2 - y^2 = 1$

2 (area sector) = t

2 (area of sector) = t

Cosh x = Ave of e^x and e^{-x}



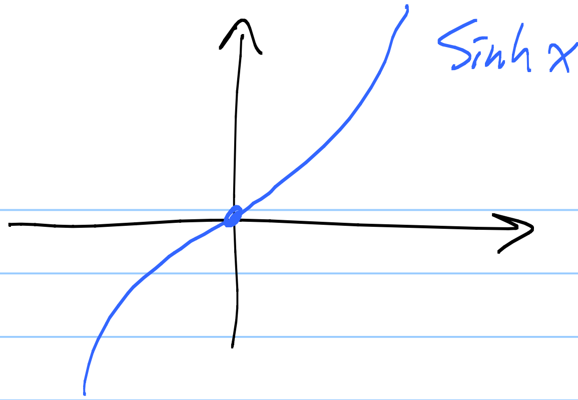
Cosh x shape of hanging cable

Facts: $\operatorname{Cosh}(-x) = \operatorname{Cosh} x$ ← Even

$$\lim_{x\to\infty} \operatorname{Cosh} x = \infty \qquad \operatorname{Cosh} x = \underbrace{\tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}}_{> \frac{1}{2}e^{x}}$$

$$\lim_{x\to-\infty} \operatorname{Cosh} x = \infty$$

$$\operatorname{Sinh}(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2} = -\operatorname{Sinh} x \leftarrow \text{Odd}$$



$$\sinh(-BIG) = \frac{e^{(-BIG)} - e^{-(-BIG)}}{2}$$

$$= \underbrace{\frac{e^{-BIG}}{2}}_{\text{small}} - \underbrace{\frac{e^{BIG}}{2}}_{\text{blows up}}$$

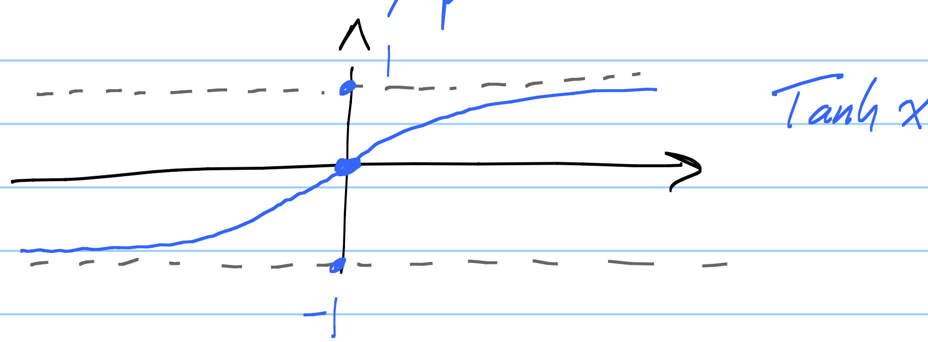
$$\lim_{x\to\infty} \sinh x = \infty$$

$$\lim_{x\to-\infty} \sinh x = -\infty$$

boss as $x \to \infty$

$$\tanh x = \frac{e^{x} - e^{-x}}{\underset{\text{boss}}{e^{x}} + e^{-x}} = \frac{\cancel{e^{x}}}{\cancel{e^{x}}} \cdot \frac{(1-e^{-2x})}{(1+e^{-2x})} \to \underbrace{\frac{1-0}{1+0}}_{1} \text{ as } x\to\infty.$$

$y=1$ is a horizontal asympt!



$$y = \sinh^{-1} x \qquad x = \sinh y$$

$$\underbrace{\frac{d}{dx}(x)}_{1} = \cosh y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$