Lesson 20 on 3.8 Exponential growth and decay

$F = ma$ $\quad F = m\dfrac{d^2y}{dt^2}$ ← Ordinary Differential Equation (ODE)

EX: $\dfrac{dy}{dx} = c$, a constant. Fact: The solution is

$$y = cx + b.$$

Population growth problem: $\dfrac{dP}{dt} = kP$ ← First order ODE

$$\underbrace{\frac{1}{P}\frac{dP}{dt}} = k$$

Aha! $\dfrac{d}{dt}(\ln P) = k$

Fact ⟹ $e^{\ln P} = e^{kt+b}$

$$P = e^{kt+b} = e^{kt}\underbrace{e^{b}}_{\substack{\text{constant}\\ \text{call it } C}} = Ce^{kt}$$

Initial condition: Know $P(0) = P_0$ ← Given.

$$P(t) = Ce^{kt}$$

Plug in $t=0$: $P(0) = Ce^{k\cdot 0} = P_0$

$C\cdot 1 = P_0$ Aha! $C = P_0$.

Sol^n to $\begin{cases} \dfrac{dP}{dt} = kP \\ P(0) = P_0 \end{cases}$ is $\boxed{P = P_0 e^{kt}}$

Prob: Pop of earth in 1950: 2.5×10^9 ← Take $t=0$ at 1950.

in 2000: 6×10^9

What will it be in 2050?

$$P = P_0 e^{kt} = (2.5 \times 10^9)\, e^{kt}$$

Also know at $t = 50$ (2000): $6 \times 10^9 = (2.5 \times 10^9)\, e^{k \cdot 50}$

$$\ln \frac{6 \times 10^9}{2.5 \times 10^9} = \ln e^{50k}$$

$$\ln \frac{6}{2.5} = 50k \qquad \boxed{k = \frac{1}{50} \ln \frac{12}{5}}$$

$$\boxed{P(t) = (2.5 \times 10^9)\, e^{\left(\frac{1}{50}\ln\frac{12}{5}\right)t}}$$

Ans: In 2050, $t = 100$: $P(100) = \underbrace{(2.5 \times 10^9)}_{1950 \text{ pop}}\, \underbrace{e^{\left(\frac{1}{50}\ln\frac{12}{5}\right)100}}_{e^{2\ln\frac{12}{5}}}$

$$e^{2\ln\frac{12}{5}} = e^{\ln\left(\frac{12}{5}\right)^2} = \left(\frac{12}{5}\right)^2 = \frac{144}{25} = 5\,{}^{19}\!/_{25}$$

Doubling time: D

$$P(t+D) = P_0 e^{k(t+D)} = \underline{P_0 e^{kt}}\, \underbrace{e^{kD}}_{2} = 2P(t)$$

$$\boxed{e^{kD} = 2} \qquad \boxed{D = \frac{\ln 2}{k}}$$

$$kD = \ln 2$$

Note: $P = P_0 e^{kt} = P_0 (\underbrace{e^k}_{a})^t = P_0 a^t \qquad (a > 0)$

Radioactive decay: $\dfrac{dm}{dt} = km$ where $k < 0$.

Write $k = -c$ where $c > 0$.

$$m = m_0 e^{kt}$$

$$\boxed{m = m_0 e^{-ct}}$$

Half-life: $m(t+H) = m_0 e^{-c(t+H)} = \underbrace{m_0 e^{-ct}}_{m(t)}\, \underbrace{e^{-cH}}_{1/2} = \frac{1}{2} m(t)$

$$e^{-cH} = \frac{1}{2}$$

$$-cH = \ln\tfrac{1}{2} = \ln 2^{-1} = -1\cdot\ln 2$$

$$\boxed{H = \frac{\ln 2}{c}} \qquad \text{Book: } H = \frac{\ln 2}{-k} \quad (k<0)$$

Prob: Cobalt-60 has a half-life of 5.24 years.

a) How much is left of a 100 mg chunk after 20 years?

b) How long until only 1 mg left?

Know $m(t) = m_0 e^{-ct}$ Given $H = 5.24 = \dfrac{\ln 2}{c}$

So $\boxed{c = \dfrac{\ln 2}{5.24}}$

$$m(t) = 100\, e^{-\frac{\ln 2}{5.24}t}$$

Given $\underline{m_0 = 100 \text{ mg}}$

a) $m(20) = 100\, e^{-\frac{\ln 2}{5.24}\cdot 20}$

$= 7.096$ mg

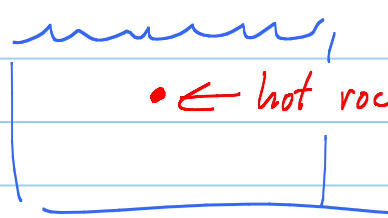
b) $100\, e^{-\frac{\ln 2}{5.24}t} = 1$ ← Solve for t.

$$\ln e^{-\frac{\ln 2}{5.24}t} = \ln\frac{1}{100}$$

$$-\frac{\ln 2}{5.24}\cdot t = \ln 100^{-1} = -1\cdot\ln\underbrace{100}_{10^2} = -2\ln 10$$

$$t = \frac{2\ln 10}{\ln 2}\cdot 5.24 = 34.8 \text{ yrs}$$

Newton's Law of Cooling:



$T(t)$ = temp at time t

T_s = Temp of surrounding ocean

$$\frac{d}{dt}\underbrace{(T-T_s)}_{P} = -k\underbrace{(T-T_s)}_{P}$$

Same ODE!

Sol^n $\quad (T(t) - T_s) = (T_0 - T_s)e^{-kt}$

$$\boxed{T(t) = T_s + (T_0 - T_s)e^{-kt}}$$

Prob; A 60° stiff was found in a 40° corn field. One hour later, it was 55°. What was the time of death?

Given $T_s = 40$. Take $t=0$ at 60° point. $T_0 = 60^\circ$

$$T(t) = 40 + (60-40)e^{-kt}$$

$$\boxed{T(t) = 40 + 20e^{-kt}}$$

But $\quad T(1) = 40 + 20e^{-k\cdot 1} = 55$ Given.

$$20e^{-k} = 15$$

$$e^{-k} = \frac{15}{20} = \frac{3}{4}$$

$$-k = \ln\frac{3}{4} = -\ln\frac{4}{3}$$

$$\boxed{k = \ln\frac{4}{3}}$$

$$T(t) = 40 + 20e^{-(\ln\frac{4}{3})t} = 98.6$$

$$20e^{-(\ \ )} = 58.6$$

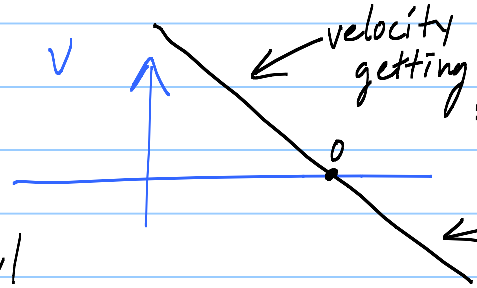
$$e^{-(\ \ )} = \frac{58.6}{20}$$

$$-(\ \ ) = \ln\frac{58.6}{20}$$

$$t = \frac{-\ln\frac{58.6}{20}}{\ln\frac{4}{3}} = -3.74 \text{ hrs.}$$

Time of death 3.74 hrs before 60° point

Speeding up



velocity getting smaller
slowing down

getter more and more negative!
speeding up.

speed = |velocity|