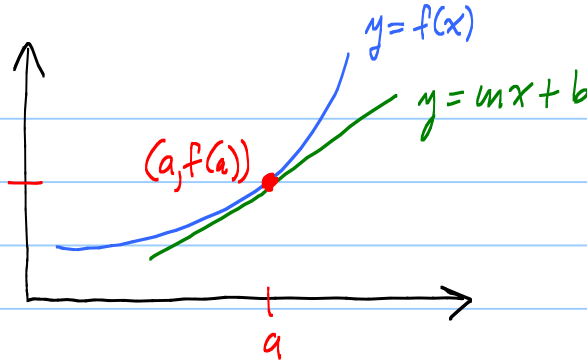
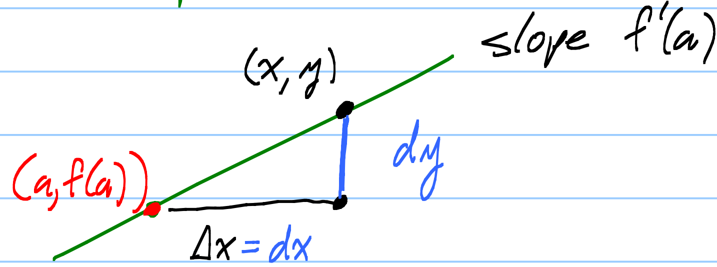
Lesson 23 on 3.10 Linear approximations



$m = \text{slope} = f'(a)$



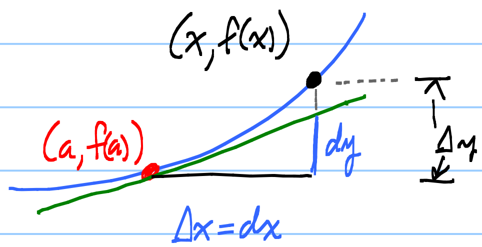
$$\frac{dy}{dx} = f'(a)$$

$$\frac{y - f(a)}{x - a} = f'(a)$$

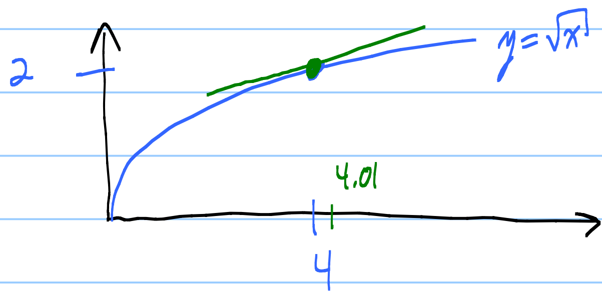
$$\boxed{dy = f'(a)\,dx}$$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

Big Idea: Near a, $\Delta y \approx dy \leftarrow$ differential of y



EX: Use tangent line to approx $\sqrt{4.01}$



$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f(4) = \sqrt{4} = 2$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

Tang line: $y = f(a) + f'(a)(x-a)$

$y = 2 + \left(\frac{1}{4}\right)(x-4)$

Near 4: $\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$

$\sqrt{4.01} \approx 2 + \frac{1}{4}(4.01-4)$

$2 + \frac{1}{400} = \underline{2.0025}$

Actual value 2.002498439...
Good to 4 decimal places!

EX: Approximate $\ln .99$ Key: Know $\ln 1 = 0$ ($a = 1$)

$f(x) = \ln x$ | $f(a) = f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x}$ | $f'(1) = \frac{1}{1} = 1$

$$y = f(a) + f'(a)(x-a) = 0 + (1)(x-1)$$

Near 1: $\ln x \approx x - 1$

$\ln .99 \approx (.99) - 1 = -0.01$

Actual value $-.0100503\ldots$

EX: $f(x) = \sin x$ near $x = 0$.

$f'(x) = \cos x$

$$y = f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$$

$\sin x \approx x$ when x near zero.

Of course! $\frac{\sin x}{x} \to 1$ as $x \to 0$

WebAssign $e^{-0.015}$ e^x $e^0 = 1$ $a = 0$

$\sqrt{99.8}$ $\sqrt{x}$ $\sqrt{100} = 10$ $a = 100$

Prob: How much paint to paint a spherical water tank $R = 15m$?

Paint is $1\text{ mm} = 10^{-3}\text{ m}$ thick. $V = \frac{4}{3}\pi R^3$

$$\text{Volume of paint} = \Delta V = \frac{4}{3}\pi(15 + 10^{-3})^3 - \frac{4}{3}\pi 15^3$$

$\boxed{dy = f'(x)\,dx}$

$$\Delta V \approx dV = \frac{dV}{dR}\,dR$$

$$= \left(\frac{4}{3}\pi 3R^2\right) dR$$

$$\Delta V \approx 4\pi R^2\, dR$$

$$4\pi(15)^2\, 10^{-3} = \frac{900\pi}{1000} = \frac{9}{10}\pi$$

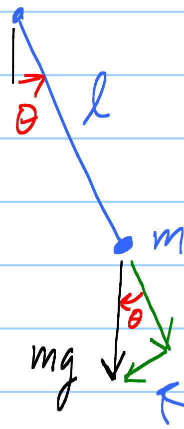
Hmmm. If I divide the volume of the paint by the thickness of the paint, I'd get the surface area of sphere.

Fact: Surface area $= \dfrac{dV}{dR}$

$$dV = \frac{dV}{dR}\, dR$$

Now divide by "thickness" dR

Who cares?



$$F = ma$$

$$-mg\sin\theta = m\ell\ddot{\theta}$$

Force

$$\boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta}$$

Non-linear 2nd order ODE!

But, for small θ, $\sin\theta \approx \theta$.

Linearized ODE: $\boxed{\ddot{\theta} = -\frac{g}{\ell}\theta}$

$$\frac{dP}{dt} = kP$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

Sol^n $\quad \theta(t) = c_1\cos\sqrt{\frac{g}{\ell}}\,t + c_2\sin\sqrt{\frac{g}{\ell}}\,t$

$$= A\cos\left(\sqrt{\frac{g}{\ell}}\,t + \varphi\right) \leftarrow \text{Good sol}^n \text{ for small } \theta!$$

Future:

$$y = f(a) + f'(a)(x-a) + c(x-a)^2 \qquad y(a) = f(a)$$

$$y' = 0 + f'(a) + 2c(x-a) \qquad y'(a) = f'(a)$$

$$y'' = 0 + 0 + 2c$$

Aha! Need $2c = f''(a)$

$c = \frac{f''(a)}{2}$

Best quadratic approx near a is

$$y = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Taylor: Keep going.

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Same value as f at a and same derivatives up to order n.

Mind blowing fact $\quad e^x \approx 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$

$a=0$

Power series. $\quad e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$

!

Good for calculators!