Lesson 25 on 4.2 Mean Value Theorem

Extreme Value Theorem (EVT): If f is continuous on $[a,b]$, then f assumes a maximum and a minimum on $[a,b]$.

Meaning, there are c_{min} and c_{max} in $[a,b]$ such that

$$f(c_{min}) \le f(x) \le f(c_{max}) \quad \text{for all } x \in [a,b].$$

Consequently:

$f'(c_j)=0$ or DNE

	x	$f(x)$
end pts $\rightarrow$	a	$f(a)$
crt #s $\rightarrow$	c_1	$f(c_1)$
	c_2	$f(c_2)$
	c_3	$f(c_3)$
end pts $\rightarrow$	b	$f(b)$

Max and min show up in list.

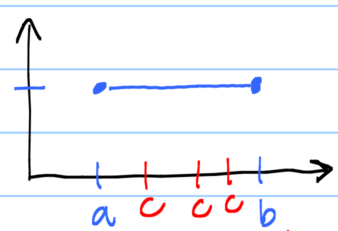
Why: EVT says the max is assumed. If it happens at c where $f'(c)$ exists, then Fermat's says $f'(c)=0$. The only other points to check are where f' DNE and endpoints.

Rolle's Theorem: 3 hypotheses:

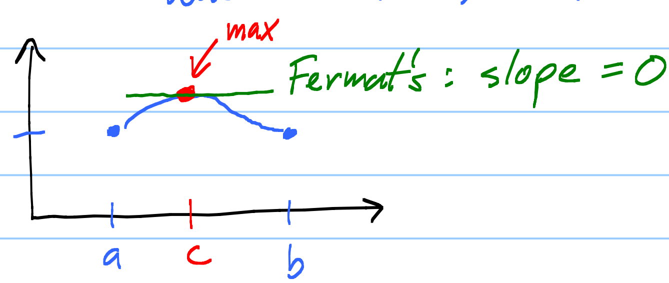
1) f continuous on $[a,b]$
2) f diff'ble on (a,b)
3) $f(a)=f(b)$

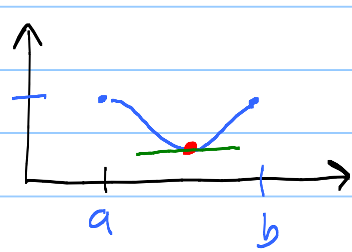
Conclusion: There exists a c with $a<c<b$ where $f'(c)=0$.

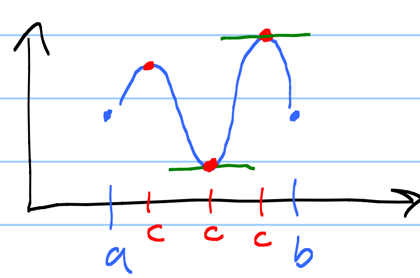
Why:



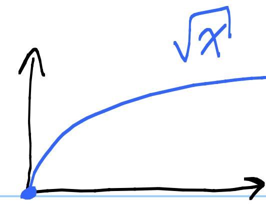
all c's work!







EX: $f(x) = \sqrt{x} - \frac{x}{2} + 3$ on $[0,4]$.



1) f cont. on $[0,4]$? Yes!

$\lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$

2) f diff'ble on $(0,4)$? Yes!

0, not diff'ble at $x=0$, but that's ok!

3) $f(0) = 0 - 0 + 3 = 3$ ✓, $f(4) = \sqrt{4} - \frac{4}{2} + 3 = 3$ ✓ $f(a) = f(b)$

Rolle's $\Rightarrow$ There is a c with $0 < c < 4$ where $f'(c) = 0$.

Find a c: $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} - \frac{1}{2} + 0 = \frac{1}{2\sqrt{x}} - \frac{1}{2} = 0$

$\boxed{c = 1}$ works.

$\frac{1}{\sqrt{x}} = 1$

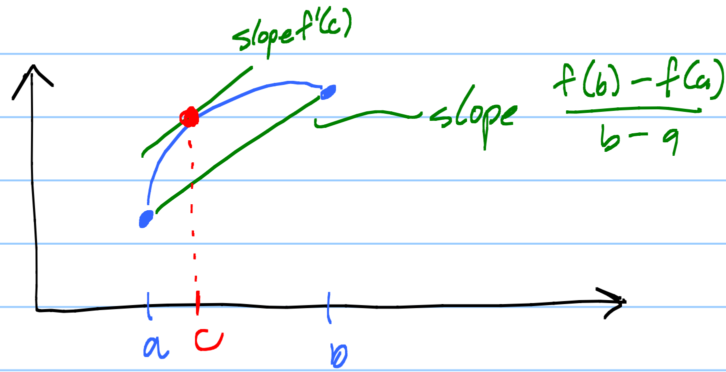
$\sqrt{x} = 1$

$x = 1^2 = 1$

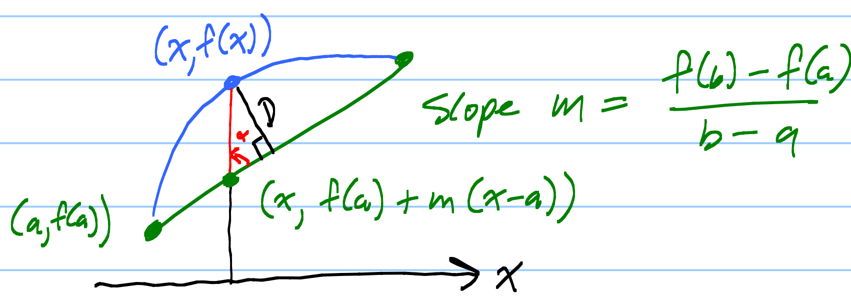
Mean Value Theorem (MVT): 2 hypothesis: 1) f cont. on $[a,b]$
2) f diff'ble on (a,b)

Conclusion: There exists a number c with $a < c < b$ where

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$



How to demonstrate:



$$D(x) = \underbrace{\sin\alpha}_{\text{const.}} \left[f(x) - [f(a) + m(x-a)] \right]$$

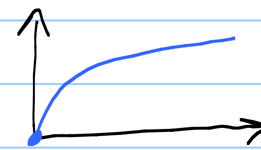
$D(x)$ satisfies hyp of Rolle's! Get c where $D'(c) = 0$.

$$D'(c) = \sin\alpha \left[f'(c) - m \right] = 0$$

Get $f'(c) = m$. ✓

EX: $f(x) = x^{2/3}$ on $[0, 8]$

1) f cont on $[0,8]$? $\quad x^{2/3} = (x^2)^{1/3}$



Yes! $\lim_{x \to 0+} x^{2/3} = 0^{2/3} = 0$

2) f diff'ble on $(0,8)$?

Yes! (Power rule)

MVT implies there is a c with $0 < c < 8$ where

$$f'(c) = \frac{f(8) - f(0)}{8 - 0} = \frac{8^{2/3} - 0}{8} = \frac{8^{2/3}}{8^1} = 8^{\frac{2}{3}-1} = 8^{-1/3}$$

$$= \frac{1}{\sqrt[3]{8}}$$

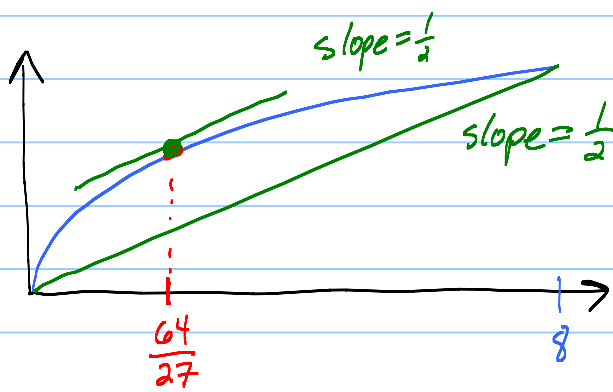
$$= \frac{1}{2}$$

or $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

Find a c:

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}} = \frac{1}{2}$$

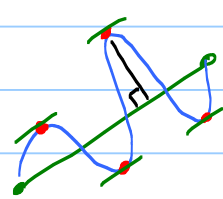


$$\frac{1}{\sqrt[3]{x}} = \frac{3}{4}$$

$$\sqrt[3]{x} = \frac{4}{3}$$

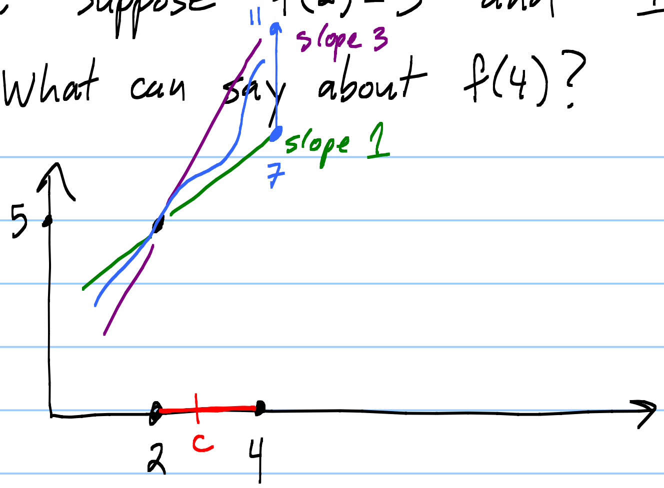
$$x = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

Can be lots of c's.



EX: Suppose $f(2) = 5$ and $1 \le f'(x) \le 3$ for all x.

What can say about $f(4)$?



MVT $\dfrac{f(4) - \overset{5}{\overbrace{f(2)}}}{4-2} = f'(c)$ between 1 and 3

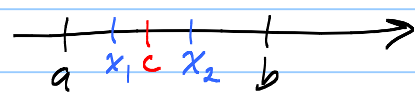
$$1 \le \frac{f(4) - 5}{2} \le 3$$

Aha! $2 \le f(4) - 5 \le 6$

$$\boxed{7 \le f(4) \le 11}$$

Big consequences of MVT:

1) If $f'(x) = 0$ on (a, b), then f is constant on (a, b)



(*) $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$ Aha! $f(x_2) = f(x_1)$.

2) If $f'(x) > 0$ on (a, b), the f is increasing

(*) $DQ = f'(c) > 0$ Aha! $f(x_2) - f(x_1) > 0$.

$f(x_2) > f(x_1)$.

3) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + C$ where C is a constant.

$[f(x) - g(x)]$ has zero derivative.

(1) $\Rightarrow$ $f - g = C$, a const.