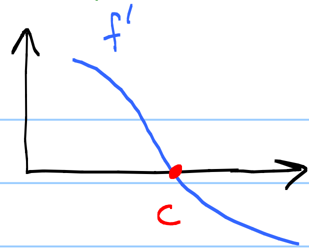
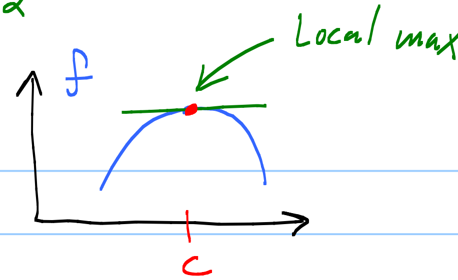
Lesson 27 on 4.3 Derivatives & graph shapes, part 2

First Derivative Test

Part 1:

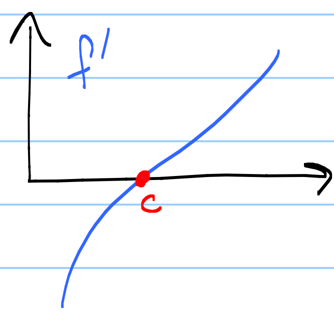


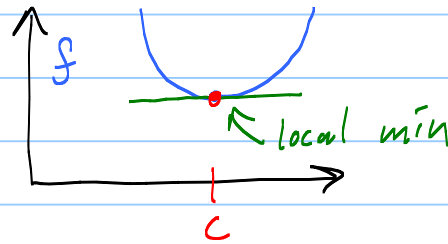
$f'(c)$, c crt #



If sign of f' goes from + to − at c, then f has a local max at c,

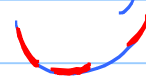
Part 2:

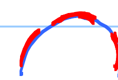




Sign f' goes from − to +, c is a local min.

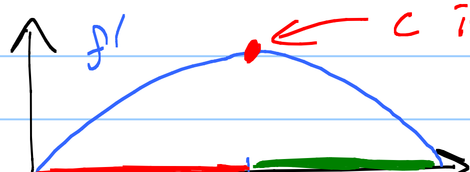
Concavity:

$f'' > 0$, Concave up  up bowl

$f'' < 0$, Concave down  down bowl

$f'' > 0$ means (f' has a positive derivative)

$f'' < 0$ means (f' has a negative derivative)



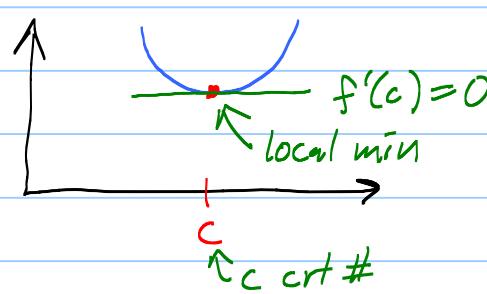
c is a pt. of inflection

$f'' > 0$ c $f'' < 0$

graph of f concave up   concave down

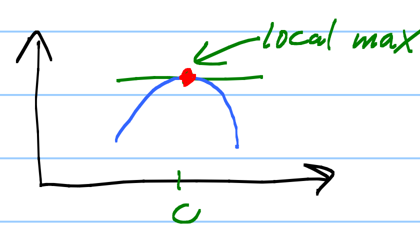
Second Derivative Test:

Part 1:

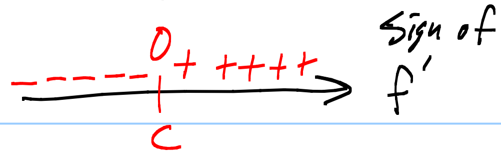


Suppose $f'(c) = 0$, If $f''(c) > 0$, then f has a local min at c.

Part 2: If $f''(c) < 0$, then f has a local max at c.



Why: $f'' = (f')'$. So $f''(c) > 0$ means f' is increasing



$f'(c) = 0$

f' must be neg to left, pos to right

First Der Test says f has a local min at c.

EX: $f(x) = \underbrace{x^2}_{u}\underbrace{\ln x}_{v}$ Find local max, min, pts of inflection

Note: only defined for $x > 0$.

Step 1: Find crt #s (where $f' = 0$).

$$f'(x) = u'v + uv' = (2x)\ln x + x^2\left(\tfrac{1}{x}\right) = 2x\ln x + x = 0$$

$$\underset{\uparrow \text{ not zero}}{x}\,(\underbrace{2\ln x + 1}_{=0}) = 0$$

$$e^{\ln x} = e^{-\frac{1}{2}}$$

$$x = e^{-1/2} = \frac{1}{\sqrt{e}}$$

Step 2: Try to use 2nd Der Test.

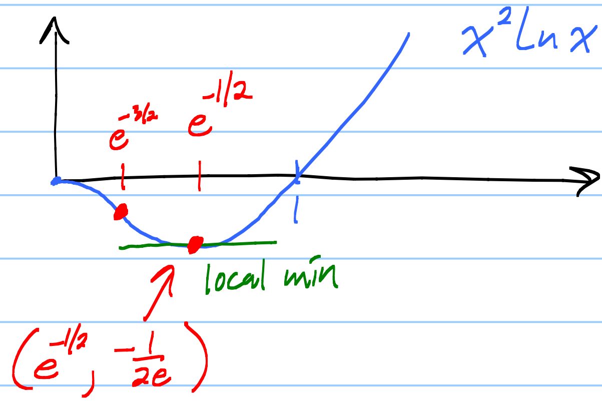
$$f''(x) = \left[2\cdot\ln x + 2x\left(\tfrac{1}{x}\right)\right] + 1$$

$$= 2\ln x + 3$$

Now $f''(c) = f''(e^{-1/2}) = 2\ln e^{-1/2} + 3 = 2\left(-\tfrac{1}{2}\right) + 3 = 2 > 0$

2nd Der Test says f has a local min at $e^{-1/2}$

Maple:



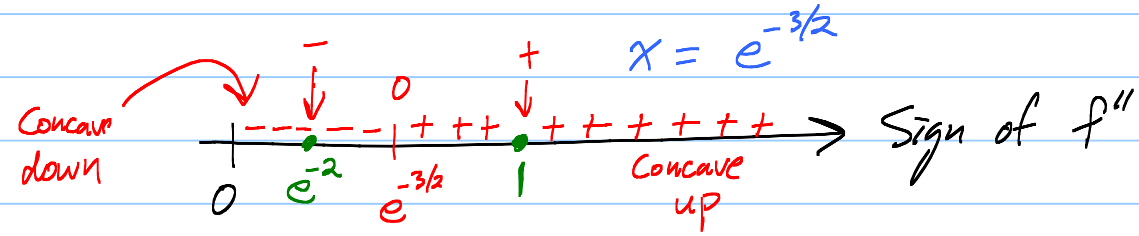
$$f(e^{-1/2}) = (e^{-1/2})^2 \ln e^{-1/2}$$

$$= e^{-1}\left(-\tfrac{1}{2}\right) = \frac{-1}{2e}$$

Concavity? $f''(x) = 2\ln x + 3 = 0$

$$\ln x = -\frac{3}{2}$$

$$x = e^{-3/2}$$



$f''(1) = 2\ln 1 + 3 = 2\cdot 0 + 3 = 3$

$f''(e^{-2}) = 2\ln e^{-2} + 3 = 2(-2) + 3 = -1$

$x = e^{-3/2}$ is a pt of inflection.

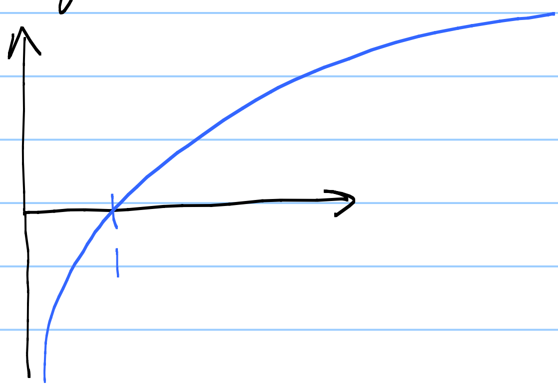
Next time: L'Hôpital's Rule shows $\lim_{x\to 0+} x^2 \ln x = 0$

$0\cdot(-\infty)$

EX: $y = \ln x \quad (x>0)$

$y' = \frac{1}{x}$ always >0 for $x>0$. increasing

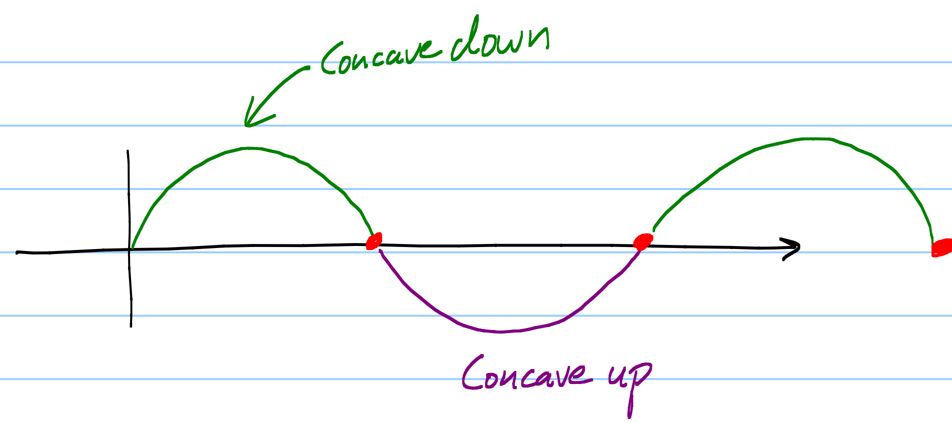
$y'' = \frac{-1}{x^2}$ always <0 for $x>0$. Concave down



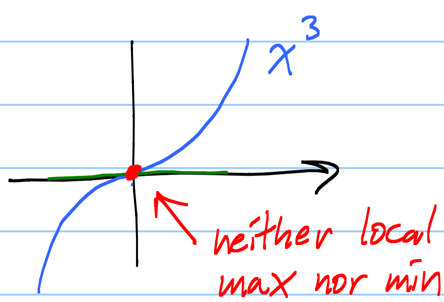
EX: $y = \sin x$

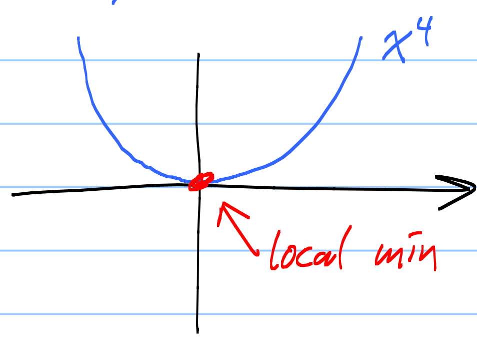
$y' = \cos x$

$y'' = -\sin x$



Note: What if $f'(c)=0$ and $f''(c)=0$ too? Test fails.



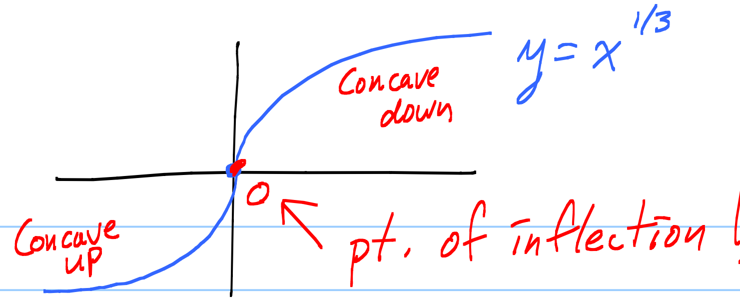


$-x^4$ has a local max at $x=0$.

Test bombs if $f''(c)=0$ too. Anything can happen!

EX: $y = x^{1/3} = g^{-1}(x)$

where $g(x) = x^3$



$y' = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3}$

always >0, but blows up at $x=0$.

$y'' = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-2/3-1} = -\frac{2}{9}x^{-5/3}$ <0 for x positive, >0 for x negative

Aha! Concavity changes at $x=0$ even though f'' DNE there.

L'Hôpital's Rule: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

EX: $\lim_{x\to 0} \frac{\tan 3x}{\sin 2x}$ $\leftarrow f$, $\leftarrow g$

$f'(x) = (\sec^2 3x)\cdot 3$

$g'(x) = (\cos 2x)\cdot 2$

$= \frac{f'(0)}{g'(0)} = \frac{(\sec^2 0)\cdot 3}{(\cos 0)\cdot 2} = \frac{1\cdot 3}{1\cdot 2} = \frac{3}{2}$

$x^2 \ln x = \frac{\ln x}{\left(\frac{1}{x^2}\right)}$ $\quad \frac{-\infty}{\infty}$ as $x \to 0+$