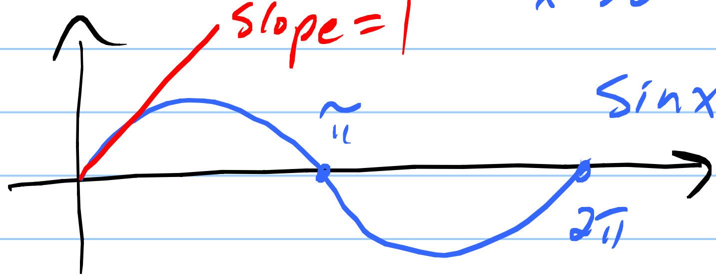
Lesson 7 on 2.5 Continuity

Famous limit: $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ $\quad \frac{0}{0}$ Ind. Form



$DQ = \frac{\sin x - \sin 0}{x - 0} = \frac{\sin x}{x}$

Radians chosen to make this slope = 1.

Degrees make slope = $\frac{\pi}{180}$!

$\frac{\sin x}{x}$

$\frac{1}{x} \sin x$

Define $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ "Continuous at $x = 0$"

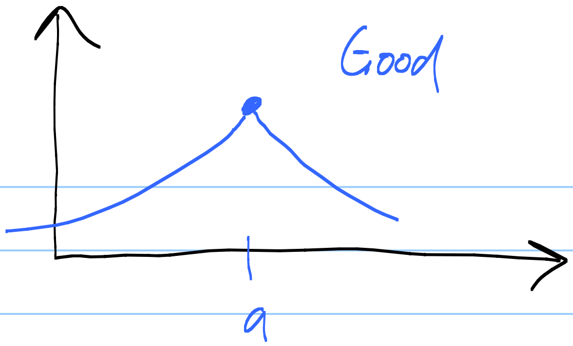
Def^n: f is continuous at $x = a$ means

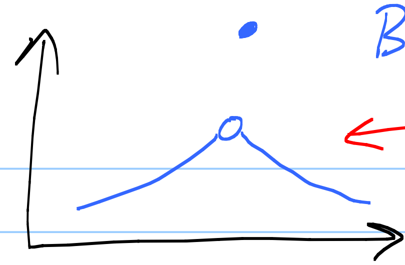
$$\lim_{x \to a} f(x) = f(a).$$

Note: Being continuous at $x = a$ means

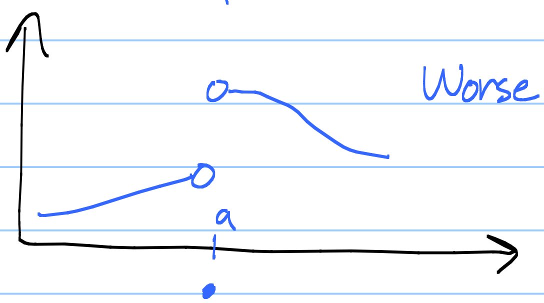
1) the limit exists, and
2) the limit $= f(a)$

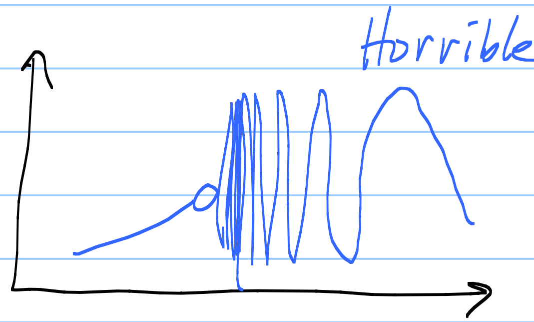
same as saying left and right limits exist and are both $f(a)$.



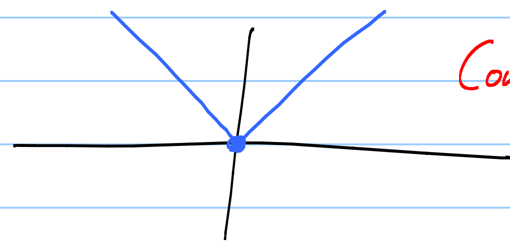


← Removable by giving it a better value!





EX: $|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$



Cont. at $x=0$.

$\lim\limits_{x\to 0+} x = 0$ $\qquad$ $\lim\limits_{x\to 0-} -x = -0 = 0$

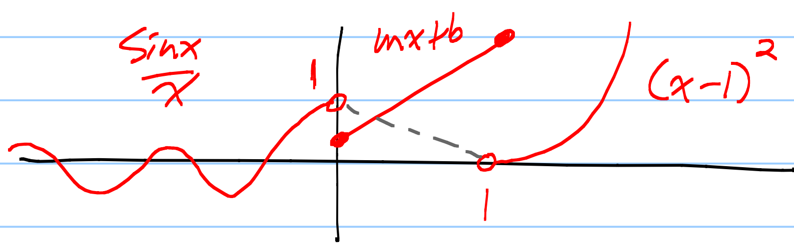
$=$

So $\lim\limits_{x\to 0} |x| = 0 = |0|$

Prob: Find constants m and b so that

$$f(x) = \begin{cases} \frac{\sin x}{x} & x < 0 \\ mx+b & 0 \leq x \leq 1 \\ (x-1)^2 & x > 1 \end{cases}$$

is continuous at all points.

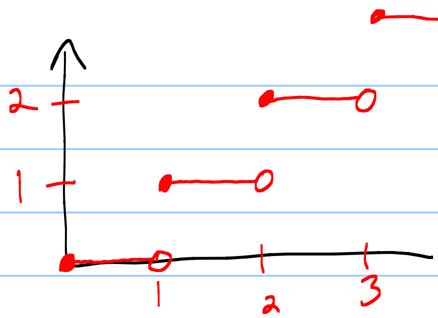


$b = 1$

Aha! Need them to hook up at jumps: $\begin{cases} m\cdot 0 + b = 1 \\ m \cdot 1 + b = 0 \end{cases}$

$m = -1$

EX: $[\![x]\!]$ = largest n such that $n \le x$.

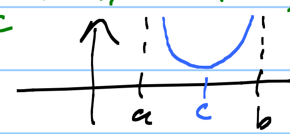


Jump discontinuities at each n.

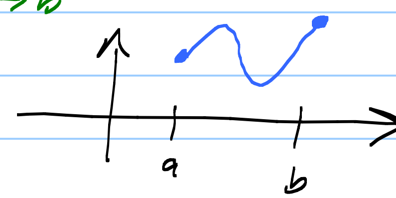
Notice that $[\![x]\!]$ is "continuous from the right".

Def^n: $\lim\limits_{x \to a+} f(x) = f(a)$. Similarly for left.

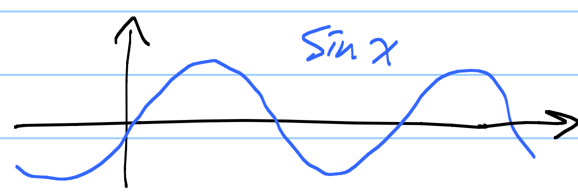
Def^n: "f continuous on (a,b)" means that $\lim\limits_{x \to c} f(x) = f(c)$ for each c in (a,b).



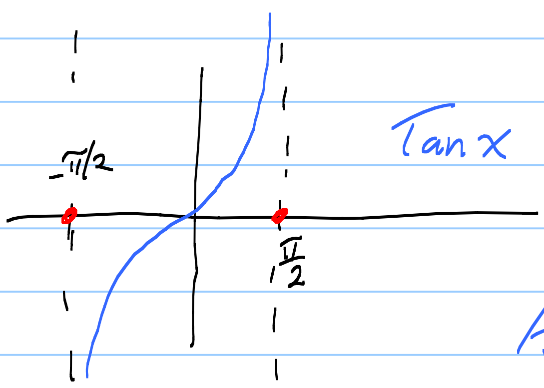
Def^n: "f continuous on $[a,b]$" means f is continuous on (a,b) and $\lim\limits_{x \to a+} f(x) = f(a)$, $\lim\limits_{x \to b-} f(x) = f(b)$



Facts: Trig fcns are continuous on their domains.



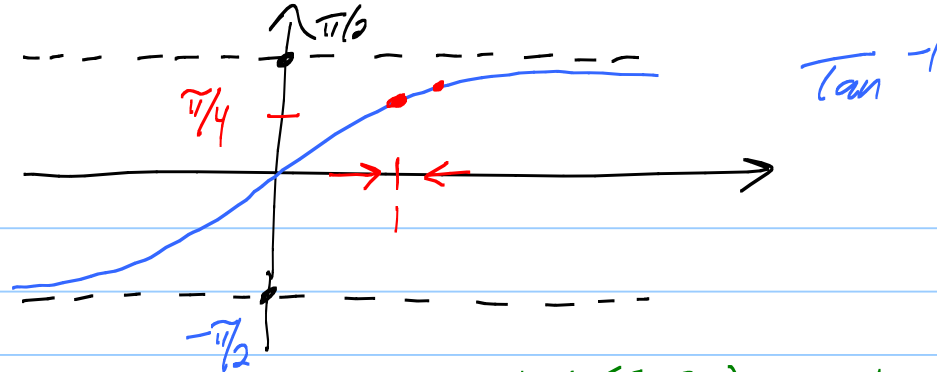
$\sin x$ cont. on $(-\infty, \infty)$



$\operatorname{Tan} x$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Also true for polys, rational functions, exponentials, log fcns, and inverse trig fcns.

EX: Find $\lim\limits_{x \to 0} \operatorname{Tan}^{-1}\left(\frac{\sin x}{x}\right) = \operatorname{Tan}^{-1}(1) = \frac{\pi}{4}$



As $\frac{\sin x}{x}$ slides into 1, $\tan^{-1}\left(\frac{\sin x}{x}\right)$ slides into $\tan^{-1}(1) = \frac{\pi}{4}$ because of continuity of $\tan^{-1}$.

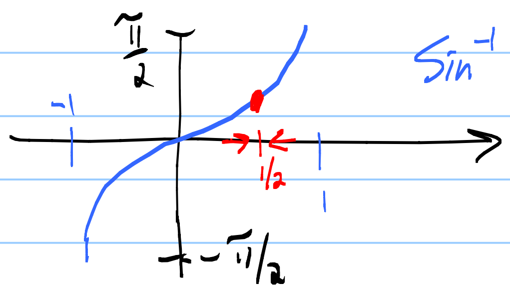
<u>Prob</u>: $\lim\limits_{x\to 1} \sin^{-1}\left(\underbrace{\frac{1-\sqrt{x}}{1-x}}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\frac{0}{0}$ at $x=1$.

$$\frac{1-\sqrt{x}}{1-x}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}} = \frac{1-(\sqrt{x})^2}{(1-x)(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}} \to \frac{1}{1+\sqrt{1}}$$

$1-(\sqrt{x})^2 \leftarrow 1-x$

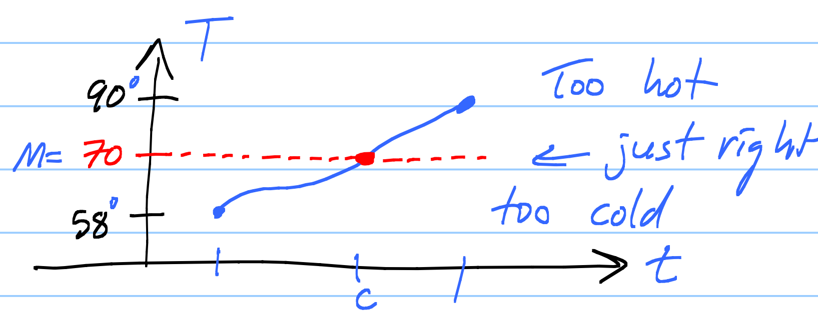
$= \frac{1}{2}$ as $x \to 1$.



<u>Fact</u>: If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a and

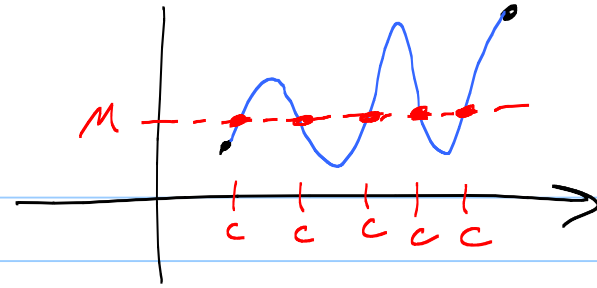
$$\lim_{x\to a} f(g(x)) = f(g(a)).$$

<u>Intermediate Value Theorem</u>

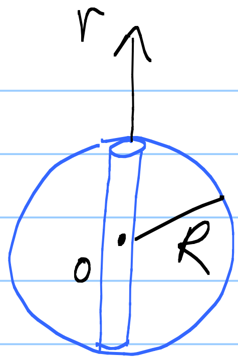


Suppose f is continuous on $[a,b]$ and $f(a) < M < f(b)$. Then there is c with $a < c < b$ such that $f(c) = M$.

Note: Can have multiple c's.

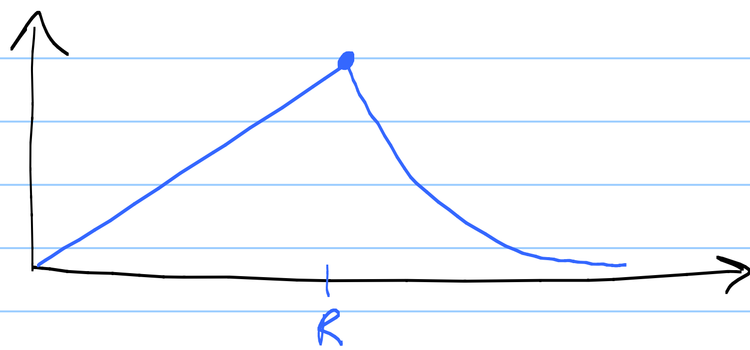


Earth:



Force $F(r) = \left(\frac{GMm}{R^3}\right) r$ inside earth (Hook's Law)

$= \frac{GMm}{r^2}$ outside earth



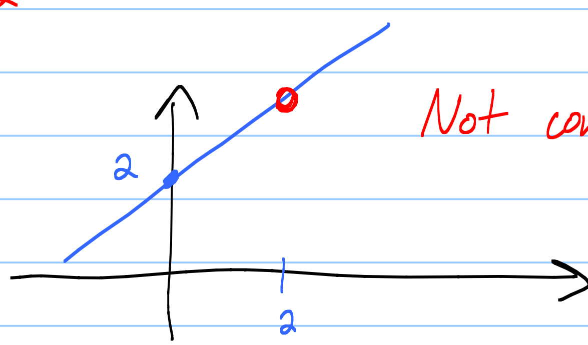
Force is continuous!

$$\lim_{r \to R^-} \left(\frac{GMm}{R^3}\right) r = \frac{GMm}{R^2} = \lim_{r \to R^+} \frac{GMm}{r^2}$$ ✓

Beware: $\frac{x^2-4}{x-2}$ ← not defined at $x=2$. $\frac{0}{0}$

Can't be continuous at $x=2$ if not defined there!

$$\frac{(x-2)(x+2)}{x-2} = x+2$$



Not continuous!

But $f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$

is.