e^x : The function $f(x)$ such that

$$f'(x) = f(x) \quad \text{and} \quad f(0) = 1.$$

Fact: If $f'(x) = k f(x)$, then

$$f(x) = Ce^{kx} \quad \text{where} \quad C = f(0).$$

EX: Let $y = f(x) = e^{a+x} = e^u$ where $u = a+x$.

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot 1 = e^{a+x}$$

Aha! $f'(x) = f(x)$. So $\underbrace{f(x)}_{e^{a+x}} = \underbrace{C}_{e^a} e^{1\cdot x}$ $= e^a e^x$

where $C = f(0) = e^{a+0} = e^a$.

Wow! $e^{a+x} = e^a e^x$

Let $x = -a$: $\underbrace{e^{a-a}}_{e^0 = 1} = e^a e^{-a}$. See $e^{-a} = 1/e^a$.

Properties of e^x: 1) $\frac{d}{dx}(e^x) = e^x$

2) $e^0 = 1$

3) $e^x > 0$ for all x, so $\frac{d}{dx} e^x = e^x > 0$ too.

Therefore e^x is strictly increasing.

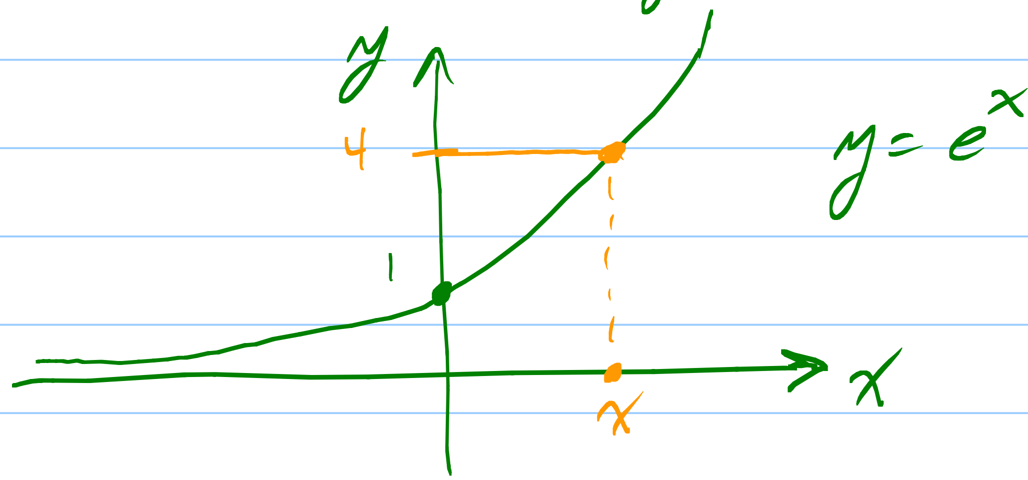
4) $\lim_{x \to -\infty} e^x = 0$, $\lim_{x \to \infty} e^x = \infty$.

5) $e^{-x} = 1/e^{x}$

6) $e^{a+b} = e^{a} e^{b}$

7) e^{x} maps $(-\infty, \infty)$ one-to-one onto $(0, \infty)$. $\mathbb{R} \to \mathbb{R}^{+}$ (It is a "group homomorphism" from the additive group of $\mathbb{R}$ to the multiplicative group of $\mathbb{R}^{+}$.)

Graph:



One-to-one. If $f(x_1) = f(x_2)$, then $x_1 = x_2$.
"Only one x has a given $f(x)$ value."

Natural log function: Ln

$\text{Ln } 4 = (x$ such tha $e^{x} = 4)$.

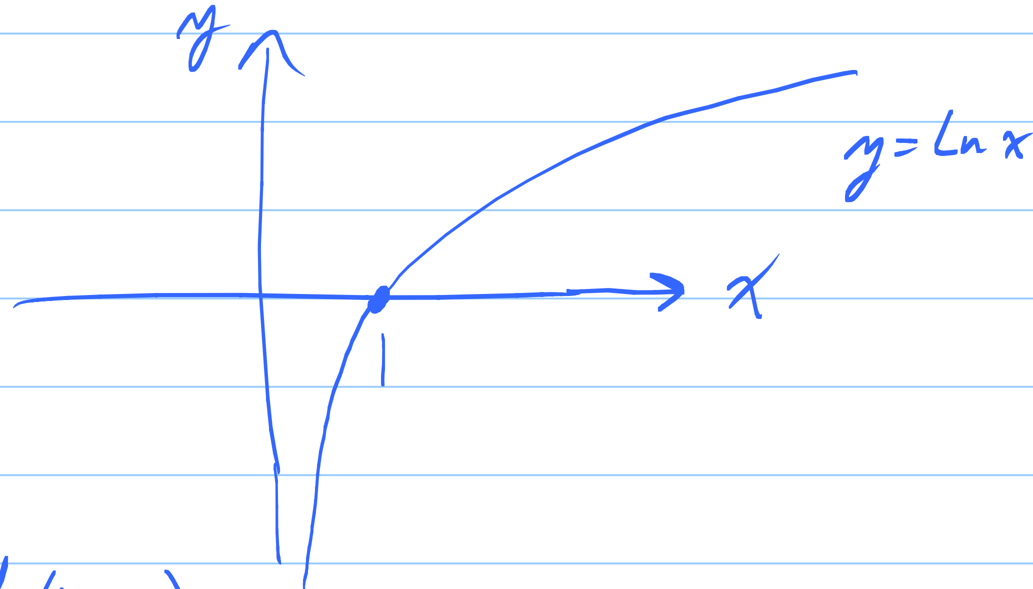
$$e^{\text{Ln} 4} = 4$$

$$f^{-1}(f(x)) = x$$

Def^n: $\ln y = \left(x \text{ such that } e^x = y\right)$.

$\ln$ is the "inverse" of e^x and vice versa.

Graph:



Prob: Find $\frac{d}{dx}(\ln x)$.

$e^{\ln x} = x$ ← diff w.r.t. x and use chain rule.

$\left[e^u \text{ where } u = \ln x\right]$

$\frac{dx}{dx} = 1$

$$\frac{d}{dx} = \frac{d}{du}\frac{du}{dx} = e^u \frac{d}{dx}(\ln x) = 1$$

$e^{\ln x} = x$

$$x \frac{d}{dx}(\ln x) = 1$$

So $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

Properties of $\ln x$: 1) $\ln 1 = 0$, $\ln e = 1$

2. $\ln(ab) = \ln a + \ln b$
3. $\ln 1/a = -\ln a$
4. $\ln a/b = \ln a - \ln b$
5. $\ln x^r = r \ln x$
6. $\frac{d}{dx} \ln x = \frac{1}{x}$

$a, b > 0$
$x > 0$
$r =$ anything.

7. $\ln x$ maps $\mathbb{R}^+$ one-to-one onto $\mathbb{R}$.
8. $\lim_{x \to 0^+} \ln x = -\infty$, $\lim_{x \to \infty} \ln x = \infty$.

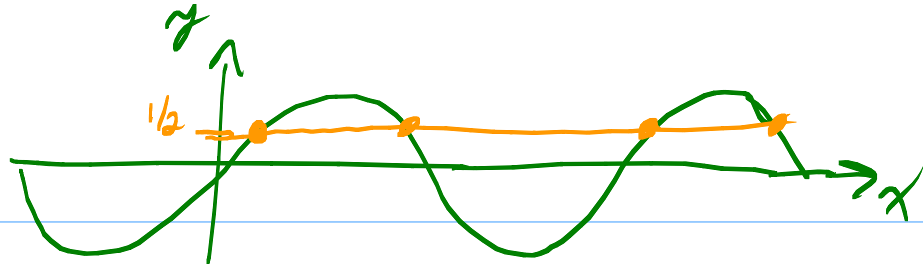
Think: e^x grows fast.

$\ln x$ grows slow.

$$\ln 2^{10} = 10 \ln 2 = 6.93\ldots$$

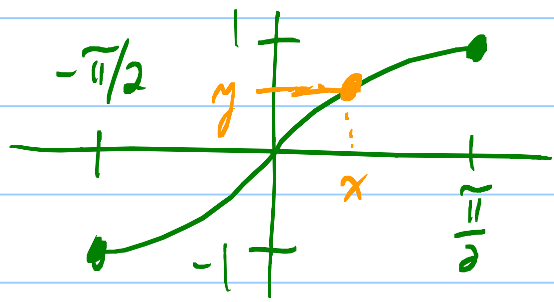
$$\ln 2^{-10} = -10 \ln 2 = -6.93\ldots$$

Inverse Trig Functions:



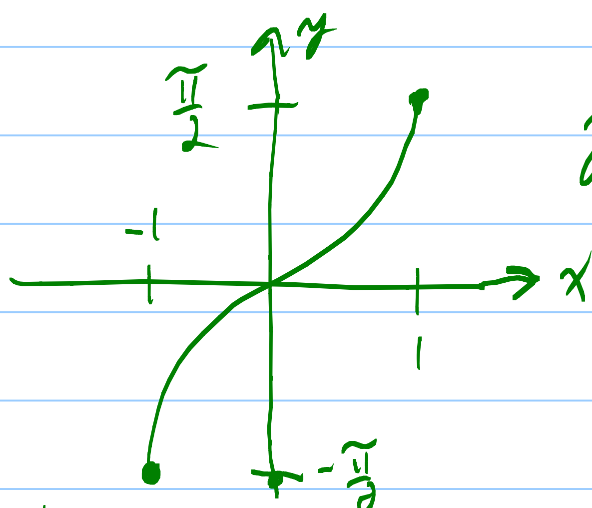
$y = \sin x$

To make $\sin x$ one-to-one, hack off the offending extra orange dots.

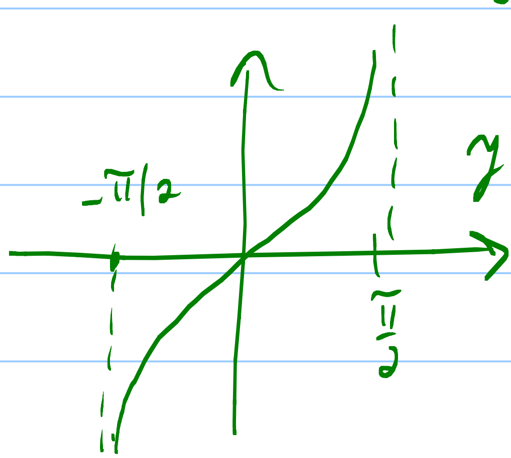


$y = \sin x$

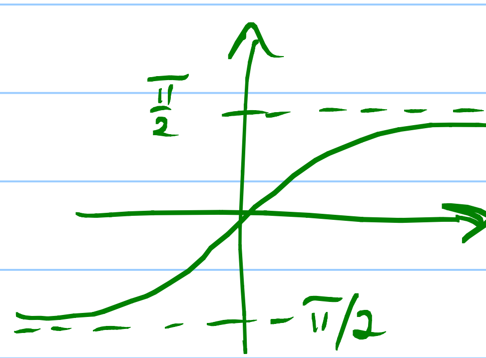
$$\underbrace{\text{ArcSin}\, y}_{\sin^{-1} y} = \left(\begin{array}{c}\text{the } x \text{ such that } \sin x = y \\ \text{with } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.\end{array}\right)$$



$y = \text{ArcSin}\, x$

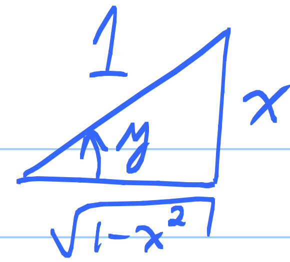


$y = \tan x$



$y = \tan^{-1} x$

Differentiate $\underbrace{y = \text{Arc Sin } x}_{x = \text{Sin } y}$



$$\frac{dx}{dx} = (\text{Cos } y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\text{Cos } y} = \frac{1}{\text{Cos}(\text{Arc Sin } x)}$$

$$= \frac{1}{\left(\frac{\sqrt{1-x^2}}{1}\right)} = \frac{1}{\sqrt{1-x^2}} \quad \checkmark$$