e^x : The y such $\frac{dy}{dx} = y$ with $y(0) = 1$.

$\sin x$: The y such $\frac{d^2y}{dx^2} = -y$ with $\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$

$\cos x$: " " " " " $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$

Fun thing: Can deduce all properties of trig functions from these.

Fact: $e^{x_1} = e^{x_2}$ implies that $x_1 = x_2$.

Show $\ln(ab) = \ln a + \ln b$

$$e^{\ln(ab)} \qquad\qquad e^{\ln a + \ln b}$$

$$\| \qquad\qquad \|$$

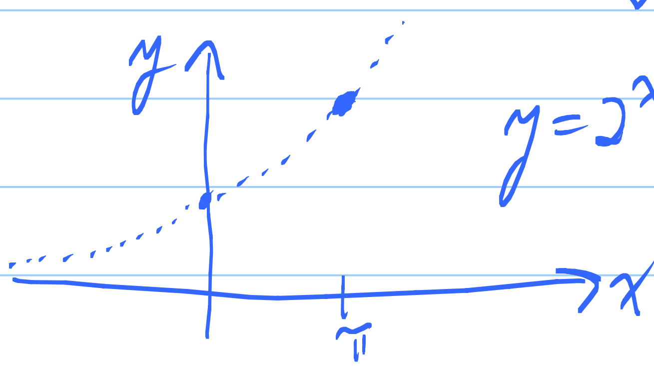
$$ab \qquad\qquad e^{\ln a} e^{\ln b}$$

$$\overset{\text{yes!}}{=} \quad a \cdot b$$

What is 2^π?

$$2^{7/8} = (2^7)^{1/8} = \sqrt[8]{2^7}$$

$$2^{3.1416} = 2^{\frac{31416}{10000}} = \sqrt[10{,}000]{2^{31416}}$$



$y = 2^x$ when x rational

$2^{\pi} = \lim_{r \to \pi} 2^r$, r rational. True, but...

$2^{\pi} = e^{\ln(2^{\pi})} = e^{\pi \ln 2}$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

3.6 Chain Rule: 1) If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))\, g'(x)$.

2) Same thing: If $y = f(u)$ where $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

EX: $y = (\sin x)^3 = u^3$ where $u = \sin x$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{dy}{dx} = (3u^2) \cdot \cos x$$

$$= 3 \sin^2 x \cos x$$

p.168: 63 $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

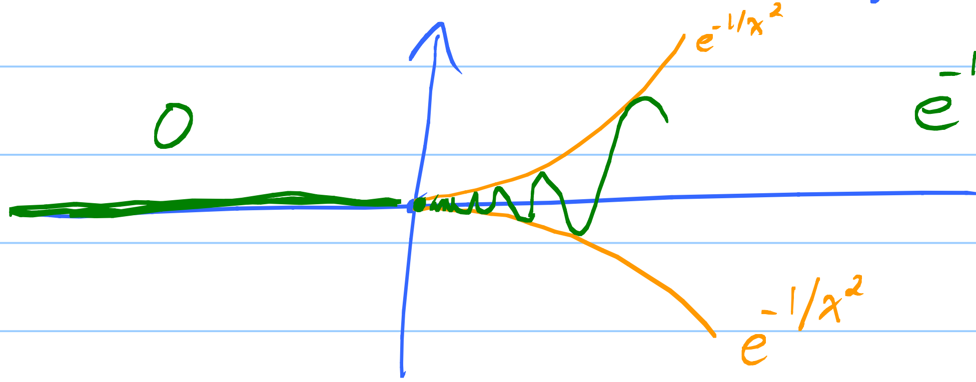
$$\frac{dy}{dx} = 3(\text{eee})^2 \left[0 + 4 (\text{mm})^3 \frac{d}{dt}[\text{mm}]\right]$$

EX: $y = (\sin(x^2+1))^5$

$$\frac{dy}{dx} = 5\left(\sin(x^2+1)\right)^4 \cos(x^2+1)\,(2x)$$

Why the Chain Rule works:

$$DQ = \frac{f(g(x)) - f(g(a))}{x-a} = \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \; \frac{g(x) - g(a)}{x-a}$$



$e^{-1/x^2} \sin \frac{1}{x}$

High School proof great if $g'(a) \neq 0$.

Honors course explanation: Write $b = g(a)$.

$$\frac{f(w) - f(b)}{w-b} = f'(b) + E(w)$$ where the

error $E(w) \to 0$ as $w \to b$. Define $E(b) = 0$

to make $E(w)$ continuous at b. Multiply by $(w-b)$:

$$f(w) - f(b) = f'(b)(w-b) + E(w)(w-b).$$

(Even true at $w=b$.)

Now let $w = g(x)$. Recall that $b = g(a)$.

$$\frac{f(g(x)) - f(g(a))}{x-a} = f'(g(a))\frac{(g(x)-g(a))}{x-a} + E(g(x))\frac{(g(x)-g(a))}{x-a}$$

Now let $x \to a$.

$$DQ \to f'(g(a)) \cdot g'(a) + 0 \cdot g'(a)$$

g diff'ble at a $\Rightarrow$ g continuous at a.

So, as $x \to a$, $g(x) \to g(a)$ and hence,

$E(g(x)) \longrightarrow E(\underbrace{g(a)}_{=b}) = 0.$

<u>EX</u>: $y = \ln(-x)$ for $x < 0$.

$= \ln u$ where $u = -x$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\cdot(-1) = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x}$$

Hmmm.

$$\begin{cases} \frac{d}{dx} \ln x = \frac{1}{x}, & x > 0 \\ \frac{d}{dx} \ln(-x) = \frac{1}{x}, & x < 0 \end{cases}$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, \quad x \neq 0.$$

Non-base e exponentials:

$$a^x = e^{\ln(a^x)} = e^{x \ln a}$$

Non-base e logs:

$$\log_a x = (y \text{ such that } a^y = x)$$

$$e^{y \ln a} = x$$

$$\ln(e^{y \ln a}) = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}$$

$$\log_a x = \frac{\ln x}{\ln a}.$$