The power rule: $\frac{d}{dx}(x^n) = n x^{n-1}$

$n = 1, 2, 3, \ldots$ $\quad$ $n = 0$

$n = -1, -2, -3, -4, \ldots$

$n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4},$

$n = p/q$ $\quad$ $x^{p/q} = (x^p)^{1/q}$ $\quad$ use chain rule

Get $\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q} x^{\frac{p}{q}-1}$

Today: The power rule works even if n is irrational.

Why: $y = x^p = e^{\ln(x^p)} = e^{p \ln x} = e^u$ where $u = p \ln x$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\left(p \cdot \frac{1}{x}\right) = \underbrace{e^{p\ln x}}_{x^p} \cdot p \cdot \frac{1}{x}$$

$$= x^p \cdot p \cdot \frac{1}{x} = p\, x^{p-1}$$

The power rule works for all $p \in \mathbb{R}$.

How to differentiate x^x: Logarithmic Diff'tion.

$$y = x^x$$

Great trick: Take the $\ln$:

$$\ln y = \ln x^x = x \ln x$$

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = 1 \cdot \ln x + x\left(\frac{1}{x}\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y\,(1 + \ln x) = x^x(1+\ln x)$$

How to see that $x^x \to 1$ as $x \to 0+$.

Trick: $\ln x^x = x \ln x = \dfrac{\ln x}{\left(\frac{1}{x}\right)}$ $\qquad \dfrac{-\infty}{+\infty}$

$$\lim_{x\to 0+} \frac{\ln x}{\left(\frac{1}{x}\right)} \overset{L'H}{=} \lim_{x\to 0+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0+}(-x) = 0$$

So $\displaystyle\lim_{x\to 0+} \ln x^x = 0$

$$x^x = e^{\ln x^x} \longrightarrow e^0 = 1 \quad \text{as } x \to 0+$$

L'Hôpital's Rule: $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

must be an

indeterminant form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

EX:
$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \quad \frac{0}{0}$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{(1+1/n)}\cdot\left(\frac{-1}{n^2}\right)}{\left(\frac{-1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

So $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$.

Finally $\left(1+\frac{1}{n}\right)^n = e^{\ln\left(1+\frac{1}{n}\right)^n} \to e^1 = e$ ✓

as $n \to \infty$. [Note: e^x is continuous at $x=1$.]

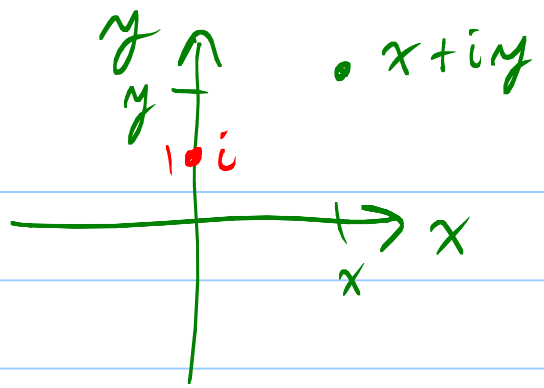
The Complex Exponential.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

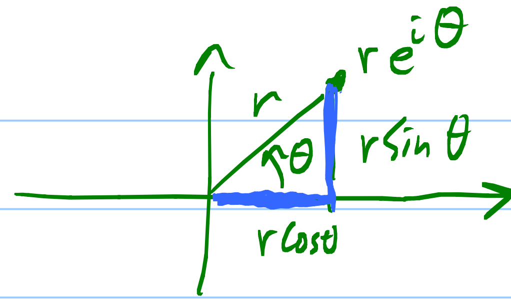
Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$ $\quad i=\sqrt{-1}$

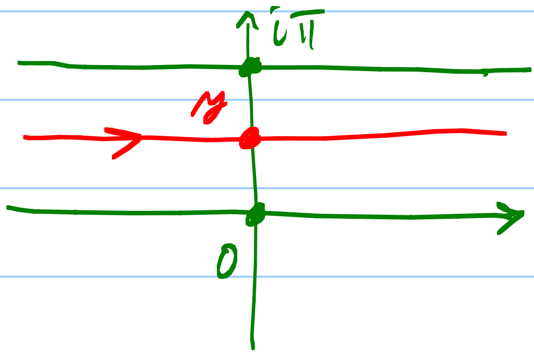
$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

$\mathbb{C}$

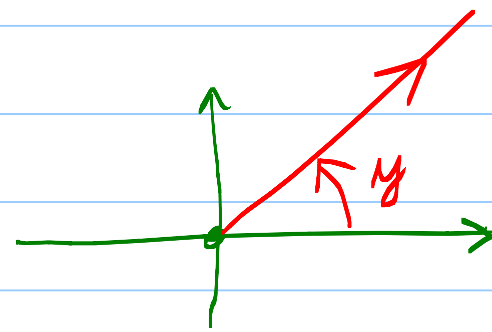


Polar form

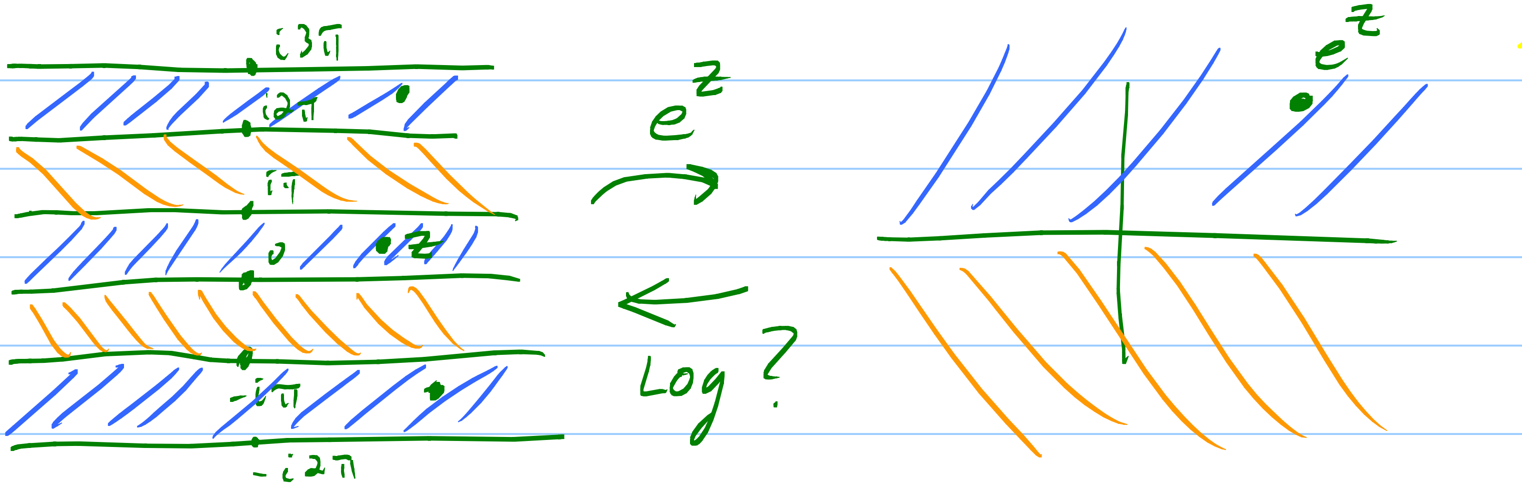




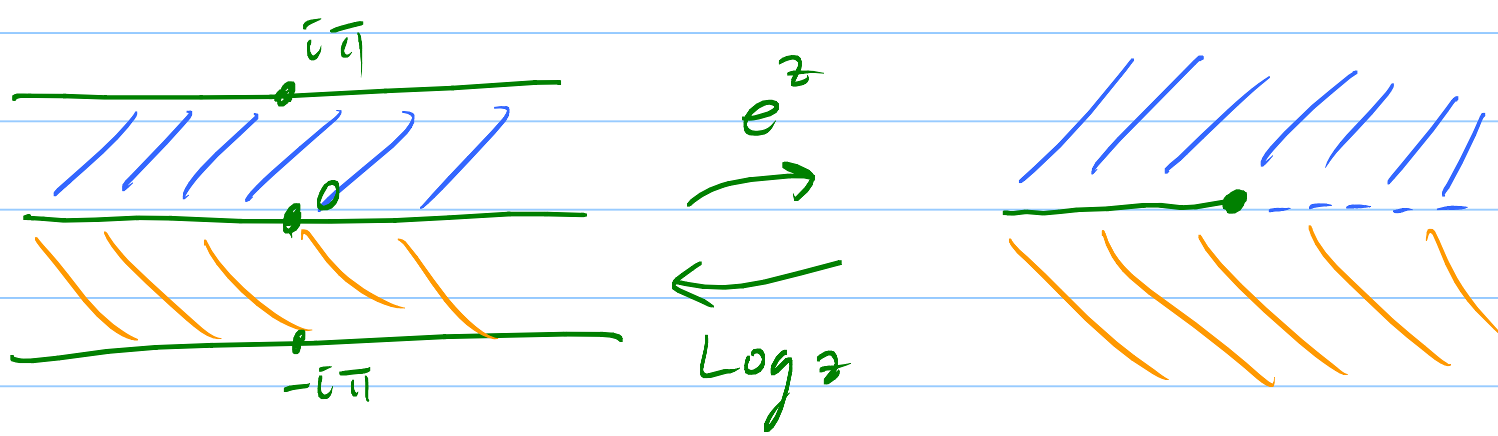
$e^{x+iy} = e^x e^{iy}$



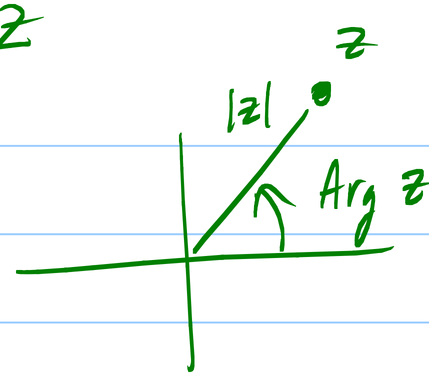
$-\infty < x < \infty$ $\qquad$ $0 < e^x < \infty$



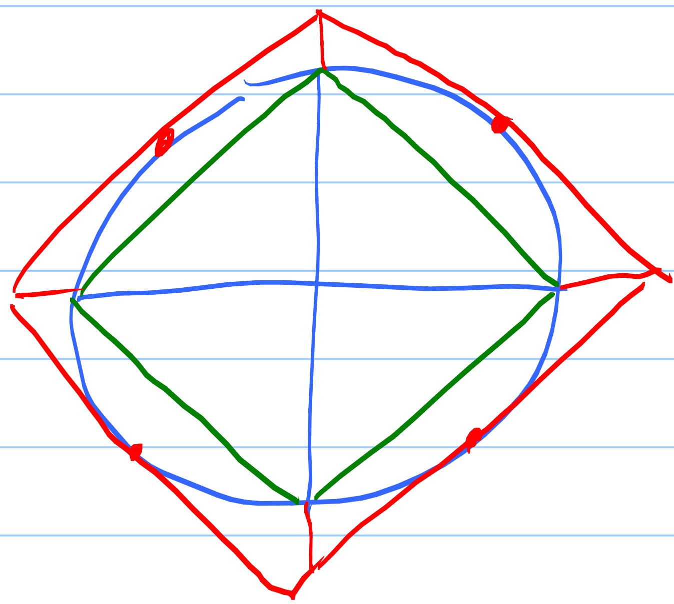
To get an inverse, hack off offending parts.



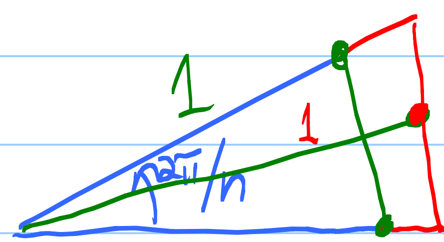
$$\operatorname{Log} z = \operatorname{Ln}|z| + i \operatorname{Arg} z$$



What is the "area" of a circle?



radius 1



$\frac{1}{2}$(Too small) $\frac{1}{2}$(Too big)