$$\frac{dy}{dx} = -2y + 4 \qquad y(0) = 3$$

Fact: If $\frac{dy}{dx} = ky$, then $y = Ce^{kx}$ where $C = y(0)$.

Just like heavy ball in ocean.

$$\frac{dy}{dx} = -2(\underbrace{y-2}_{u}) \qquad \boxed{u = y - 2}$$

$$\frac{du}{dx} = \frac{dy}{dx}$$

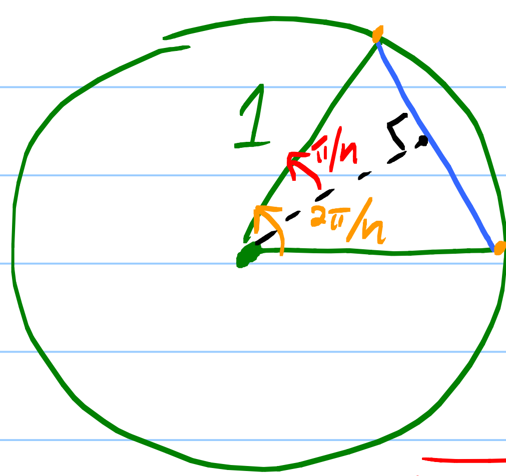
$\boxed{\frac{du}{dx} = -2u}$ So $u = Ce^{-2x}$ where $C = u(0)$

$$= y(0) - 2$$
$$= 3 - 2 = 1$$

$$u = 1 \cdot e^{-2x}$$

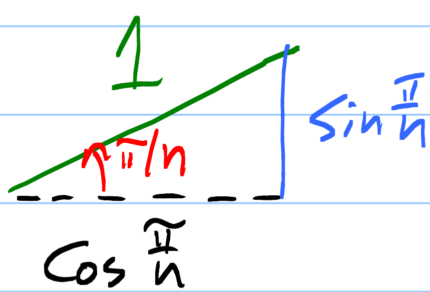
$$y = u + 2 = e^{-2x} + 2.$$

5.1 Integration.



Inner Δs

$\frac{1}{2}\Delta$



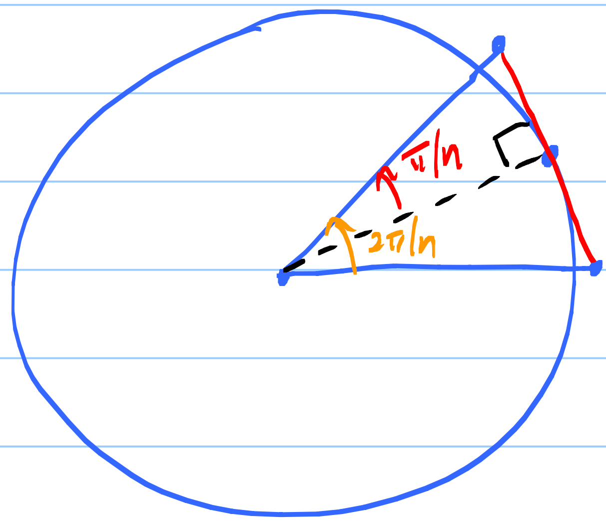
$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

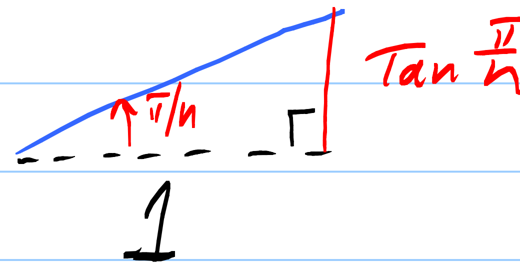
$$\text{Area of } \tfrac{1}{2}\Delta = \frac{1}{2}\cos\frac{\pi}{n}\sin\frac{\pi}{n}$$

$$\text{Area of } \Delta = \cos\frac{\pi}{n}\sin\frac{\pi}{n} = \frac{1}{2}\sin\frac{2\pi}{n}$$

Area of inscribed polygon $= \frac{n}{2} \sin \frac{2\pi}{n}$.

Outside polygon





Area of $\frac{1}{2}\Delta = \frac{1}{2} \cdot 1 \cdot \tan \frac{\pi}{n} = \frac{1}{2} \tan \frac{\pi}{n}$

Area of $\Delta = \tan \frac{\pi}{n}$

Area of circumscribed poly $= n \tan \frac{\pi}{n}$.

(Too small poly) < A < (Too big poly)

$$\frac{n}{2} \sin \frac{2\pi}{n} < A < n \tan \frac{\pi}{n}$$

$$\frac{0}{0} \qquad \frac{\sin \frac{2\pi}{n}}{\left(\frac{2}{n}\right)} < A < \frac{\tan \frac{\pi}{n}}{\left(\frac{1}{n}\right)} \qquad \frac{0}{0}$$

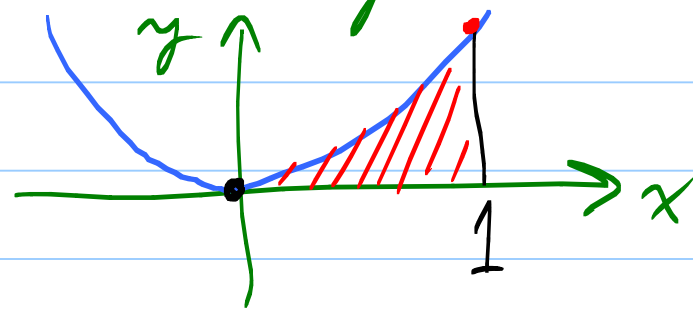
$\downarrow$ L'H $\qquad\qquad\qquad$ $\downarrow$ L'H

$$\lim_{n\to\infty} \frac{\left(\cos \frac{2\pi}{n}\right)\left(\frac{-2\pi}{n^2}\right)}{\left(\frac{-2}{n^2}\right)} \qquad\qquad \lim_{n\to\infty} \frac{\left(\sec^2 \frac{\pi}{n}\right)\left(\frac{-\pi}{n^2}\right)}{\left(\frac{-1}{n^2}\right)}$$

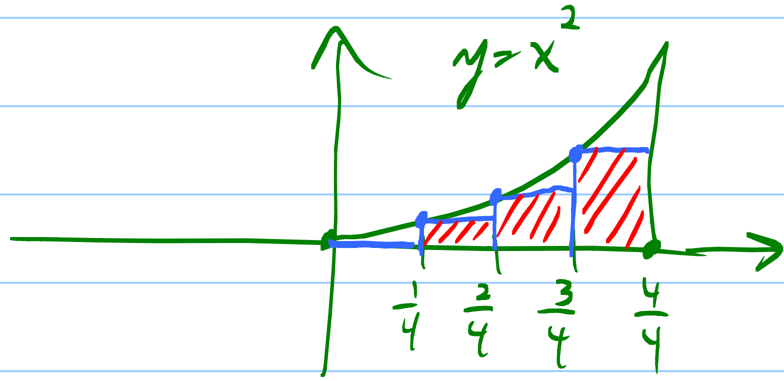
$= \pi \cdot \cos 0 = \pi$ $\qquad\qquad$ $= (\sec^2 0) \cdot \pi = \pi$

Area must be $= \pi$.

Problem: Find the area under $y = x^2$ between $x = 0$ and $x = 1$.



Too small



$$0 \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

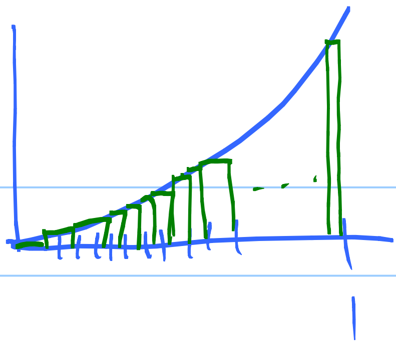
100 divisions:

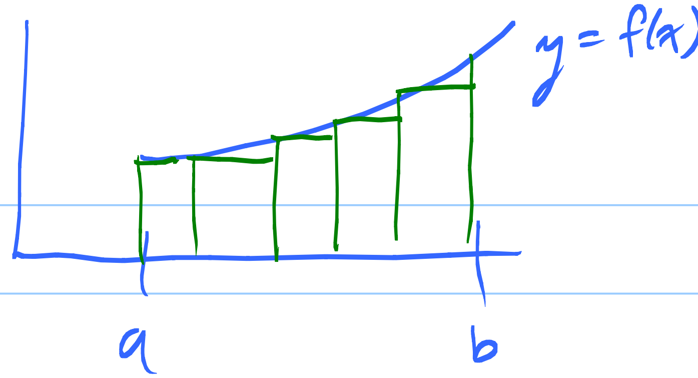
$$0 \cdot \frac{1}{100} + \left(\frac{1}{100}\right)^2 \cdot \frac{1}{100} + \left(\frac{2}{100}\right)^2 \cdot \frac{1}{100} + \cdots + \left(\frac{99}{100}\right)^2 \cdot \frac{1}{100}$$

$= \text{Sum} \quad \left(\frac{n}{100}\right)^2 \cdot \frac{1}{100}$ from $n = 0$ to $n = 99$

= sum((n/100)^2 /100, n=0..99); MAPLE

$= \sum_{n=0}^{99} \left(\frac{n}{100}\right)^2 \cdot \frac{1}{100}$ summation notation.





$\Delta x = \frac{1}{N}$ $\qquad\qquad$ $\Delta x = \frac{b-a}{N}$

Too small area

$$\sum_{n=0}^{N-1} \left(\frac{n}{N}\right)^2 \cdot \frac{1}{N} \qquad\qquad \sum_{n=0}^{N-1} f(a+n\Delta x)\cdot \Delta x$$

$$= \frac{1}{N^3}\left(0^2+1^2+2^2+3^2+\cdots+(N-1)^2\right)$$

Fact: $1^2+2^2+3^2+\cdots+M^2 = \frac{M(M+1)(2M+1)}{6}$

So $1^2+2^2+3^2+\cdots+(N-1)^2 = \frac{(N-1)(N-1+1)(2(N-1)+1)}{6}$

$$= \frac{(N-1)N(2N-1)}{6}$$

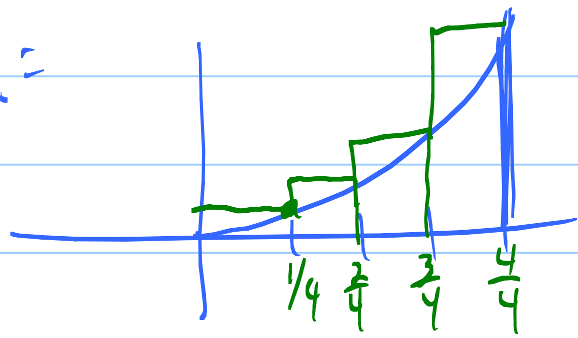
Too small area $= \frac{1}{N^3}\left[\frac{(N-1)(N)(2N-1)}{6}\right]$

$$= \frac{1}{6}\left[\frac{N-1}{N}\cdot\frac{N}{N}\cdot\frac{2N-1}{N}\right]$$

$$= \frac{1}{6}\left[\left(1-\frac{1}{N}\right)(1)\left(2-\frac{1}{N}\right)\right]$$

$$\to \frac{1}{3} \text{ as } N \to \infty.$$

Too big area:



Too big area $\left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{4}\right)^2 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{4}{4}\right)^2 \cdot \frac{1}{4}$

For N cuts: get $\sum_{n=1}^{N} \left(\frac{n}{N}\right)^2 \cdot \frac{1}{N}$

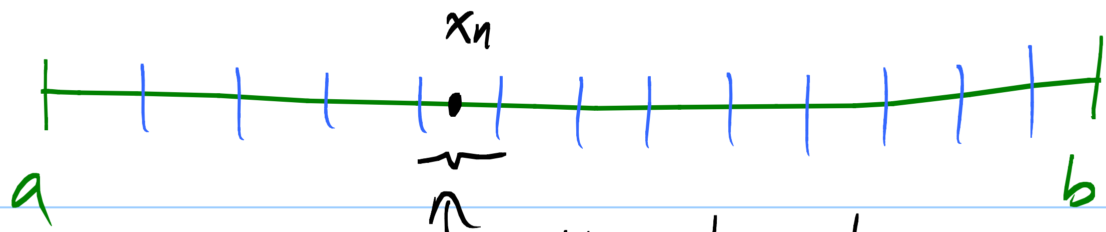
$$= \frac{1}{N^3}\left[1^2+2^2+3^2+\cdots+N^2\right] = \frac{1}{N^3}\left[\frac{N(N+1)(2N+1)}{6}\right]$$

$$\to \frac{1}{3} \text{ as } N \to \infty.$$

So actual area is $= \frac{1}{3}$.

$$\frac{1}{3} = \int_0^1 x^2\, dx = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\,\Delta x$$

where $f(x) = x^2$, $\Delta x = \frac{b-a}{N} = \frac{1-0}{N}$



n-th interval of N intervals of width $\frac{b-a}{N}$.

x_n any point in the interval. (left, right ok)