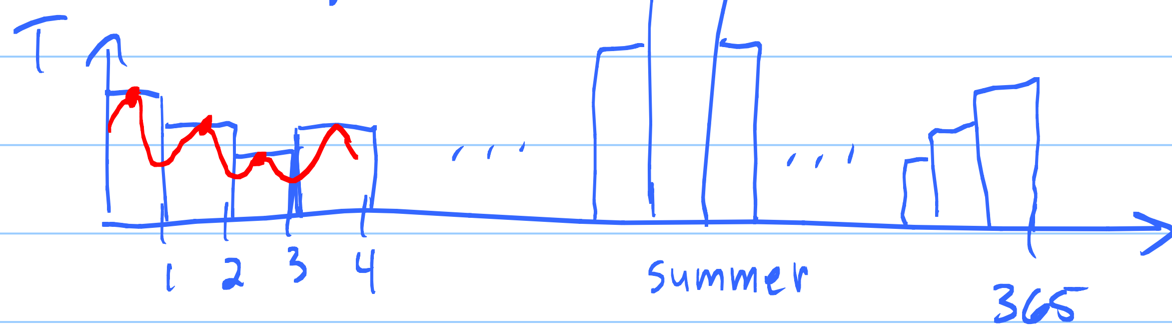
Average Value of a function

Ex: Ave High Temp

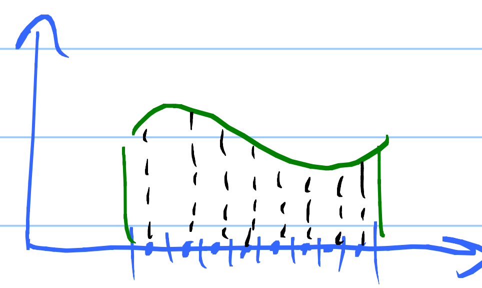


$$T_{ave} = \frac{\sum T_n}{365}$$

Claim: Average over hours, minutes, seconds.

$$T_{ave} \to \frac{\int_0^{365} T(t)\,dt}{365}$$

General $f(x)$ on $a \le x \le b$:



$$\text{Ave} \cong \frac{\sum_{n=1}^{N} f(x_n) \cdot \Delta x}{\Delta x \cdot N} \qquad \Delta x = \frac{b-a}{N}$$

$$\cong \frac{\sum_{n=1}^{N} f(x_n)\,\Delta x}{b-a} \xrightarrow[N \to \infty]{} \frac{1}{b-a} \int_a^b f(x)\,dx$$

Fact: $\text{Ave}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$

Prob: Estimate the average of $\sin(x)$ on $[0,\pi]$.

$$\text{Ave} \cong \frac{1}{\pi}\sum_{n=1}^{N} \sin\left(\frac{n\pi}{N}\right)\left(\frac{\pi}{N}\right)$$

How to find sum.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{in\pi/N} = \cos\frac{n\pi}{N} + i\sin\frac{n\pi}{N}$$

$$\boxed{\begin{array}{l}\text{Im}(a+bi) = b \\ \text{Re}(a+bi) = a\end{array}}$$

So $\sin\frac{n\pi}{N} = \text{Im}\left(e^{in\pi/N}\right) = \text{Im}\left(e^{i\pi/N}\right)^n$

$$1+r+r^2+\cdots+r^N = \frac{1-r^{N+1}}{1-r}$$

why:

$$1\cdot(1+r+\cdots+r^N) = 1+r+r^2+\cdots+r^N$$

$$-\quad r\cdot(1+r+\cdots+r^N) = r+r^2+\cdots+r^N+r^{N+1}$$

$$(1-r)(1+r+\cdots+r^N) = 1-r^{N+1}$$

$$\sum_{n=1}^{N}\sin\frac{n\pi}{N} = \sum_{n=1}^{N}\text{Im}\left(e^{i\pi/N}\right)^n$$

$$= \text{Im}\left(\sum_{n=1}^{N}(\underbrace{e^{i\pi/N}}_{r})^n\right)$$

$$= \mathrm{Im}\left[\frac{1-\left(e^{i\pi/N}\right)^{N+1}}{1-e^{i\pi/N}} - 1\right]$$

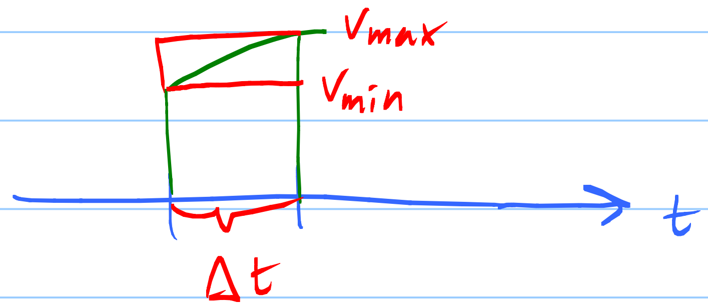
1 missing from front of Σ.

<u>Prob</u>: Sitting in train watching speedometer. How far have I gone?

$v = \frac{\Delta D}{\Delta t}$ $\quad \Delta D = v\,\Delta t$ if v is constant.

$$D \cong \sum v_n \Delta t \longrightarrow \int_a^b v(t)\,dt$$

<u>Fact</u>: $D = \int_a^b v(t)\,dt$ ← signed area under velocity curve.



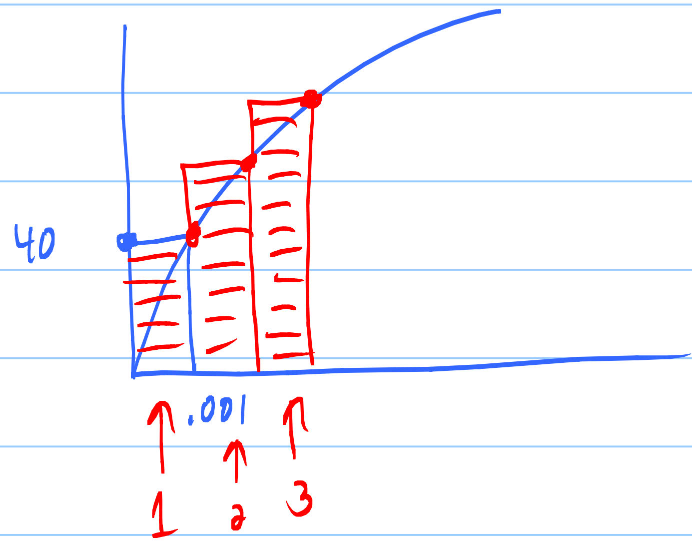
$v_{min}\,\Delta t \le dist \le v_{max}\,\Delta t$

$$\sum v_{min}\,\Delta t < \text{Actual dist} < \sum v_{max}\,\Delta t$$

As $N \to \infty$, these squeeze down to D.

p. 304 : 12

Time	Velocity	Distance
0	0	0
.001h	40 m/hr.	area 1
.002h		area 1 + 2
		area 1+2+3



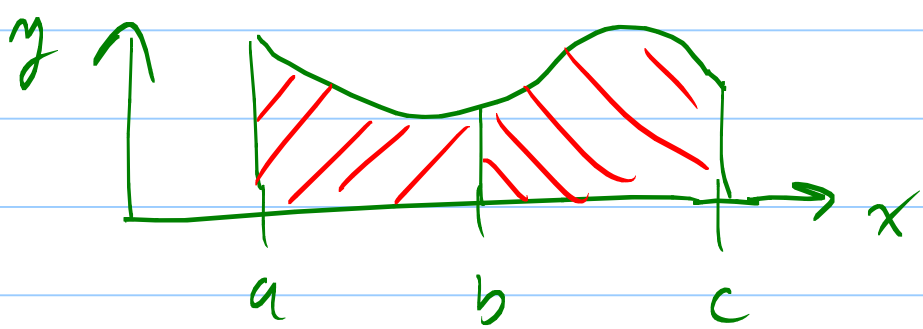
Basic Properties:

Basic Prop of Σ on p. 308

Basic Prop of $\int$ on p. 317

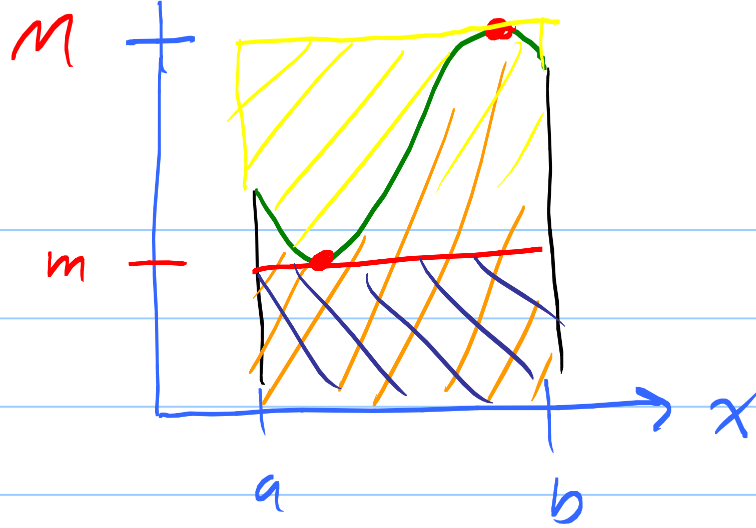
$$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$



$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Max-Min. Ineq:



$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Why: $m \le f(x_n) \le M$

$$\underbrace{\sum m\Delta x}_{m\underbrace{\sum \Delta x}_{b-a}} \le \sum f(x_n)\Delta x \le \underbrace{\sum M\Delta x}_{M(b-a)}$$

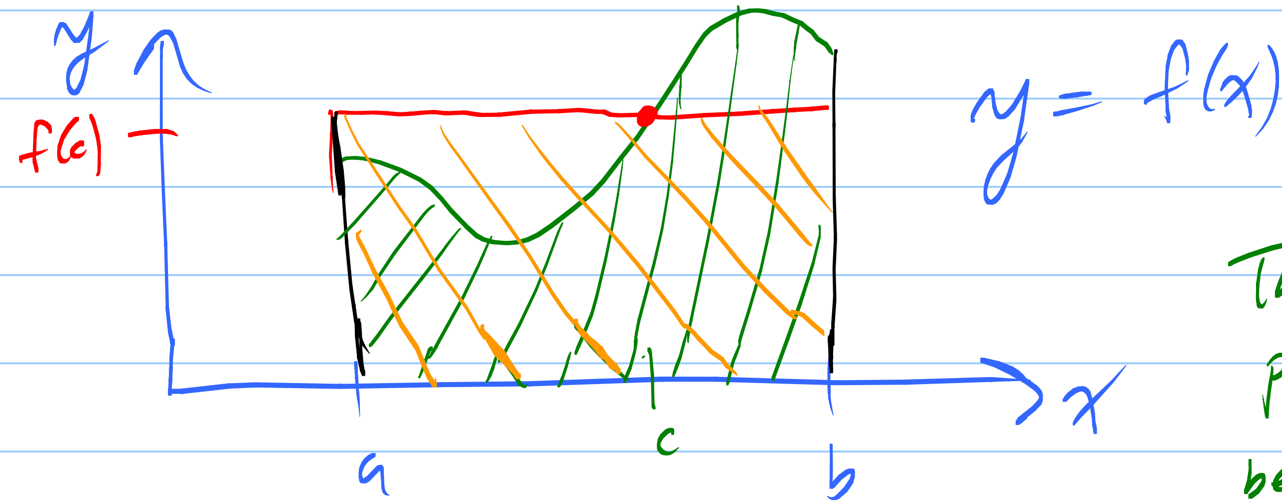
Now let $N \to \infty$.

Domination Inequality: If $g(x) \le f(x)$ on $a \le x \le b$, then
$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$$

Why: $\sum g(x_n)\Delta x \le \sum f(x_n)\Delta x$

Take limits.

Mean Value Theorem for Integrals



There is a point c between a and b such that

$$\underbrace{\int_a^b f(x)\,dx}_{\text{area under graph}} = \underbrace{f(c)\,(b-a)}_{\text{area of rectangle}}$$