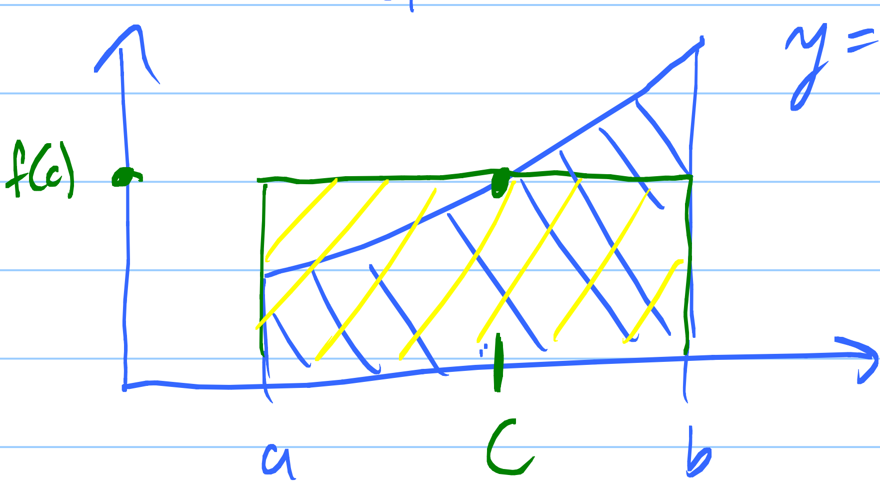
Mean Value Theorem for $\int$'s.

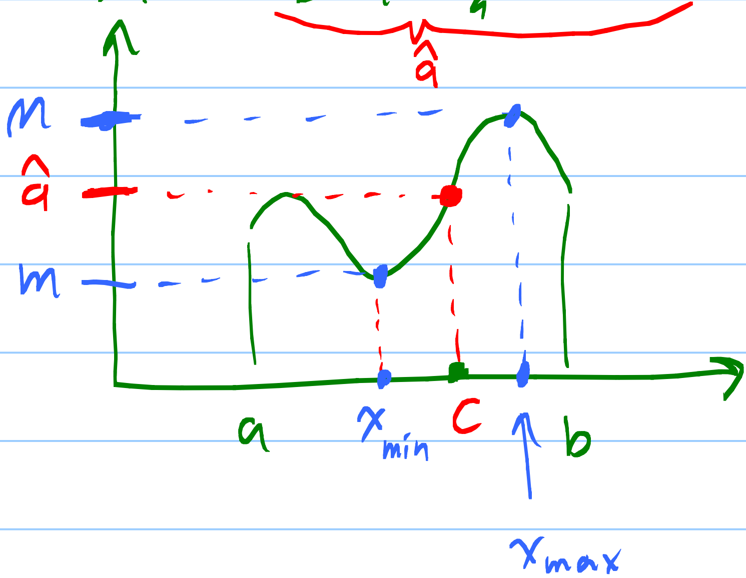
If f is continuous on $a \le x \le b$, then
$[a, b]$
there exists a point c with $a \le c \le b$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a).$$



Why it is true: $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$$m \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\hat{a}} \le M$$

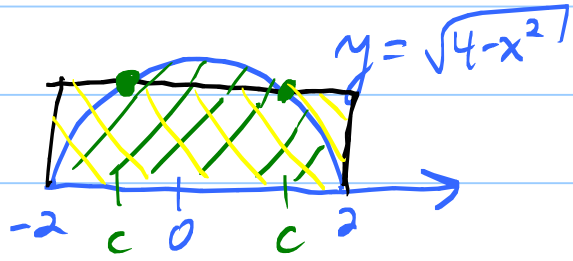


Intermediate Value Theorem: Since $\hat{a}$ is between m and M, there is a point c between x_{min} and x_{max} where $f(c) = \hat{a}$.

$$\hat{a} = \frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

EX: Find ~~the~~ a c for $\int_{-2}^{2}\sqrt{4-x^2}\,dx$

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx = \frac{1}{2}(\pi\cdot 2^2) = 2\pi$$



Want c so that

$$\text{Area} = f(c)(b-a)$$

$$2\pi = \sqrt{4-c^2}\,(2-(-2))$$

$$\frac{\pi}{2} = \sqrt{4-c^2}$$

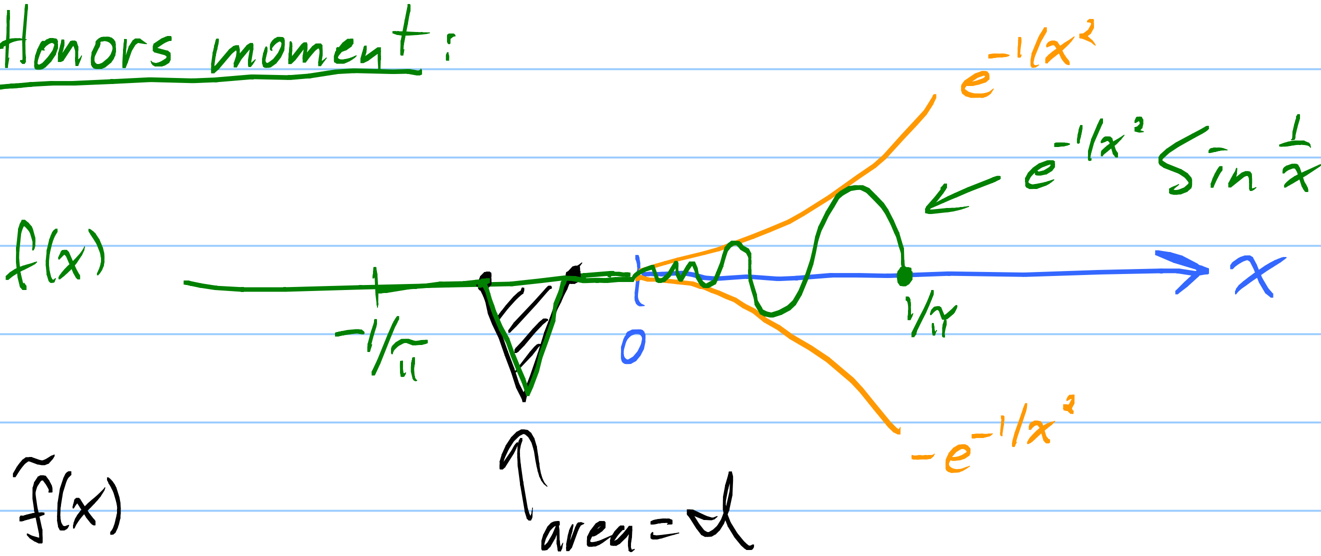
$$\frac{\pi^2}{4} = 4 - c^2$$

$$c^2 = 4 - \frac{\pi^2}{4}$$

two c's! →

$$c = \pm\sqrt{4-\frac{\pi^2}{4}}$$

Honors moment:

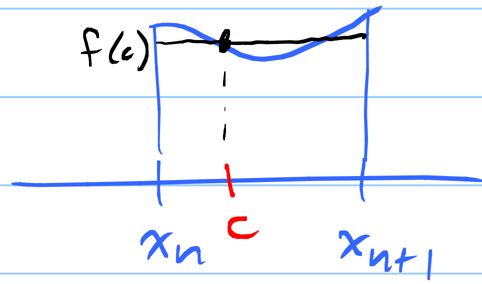


$$\int_{-1/\pi}^{1/\pi} f(x)\,dx = d.$$

$$\int_{-1/\pi}^{1/\pi} \tilde{f}(x) = -d + d = 0.$$

Now all the zeroes of $f(x)$ are good c's for the MVT for $\tilde{f}$

<u>Important use</u>:



$$\int_{x_n}^{x_{n+1}} f(x)\,dx = f(c)\underbrace{\left[x_{n+1} - x_n\right]}_{\Delta x} = f(c)\,\Delta x.$$

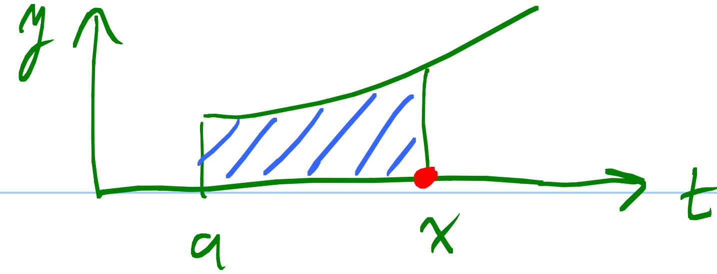
$$\int_a^b f(t)\,dt = \int_{a=x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx$$

$$= f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_N)\Delta x$$

Wow! Given an N, there is a Riemann sum that is <u>equal</u> to the integral.

<u>Fundamental Theorem of Calculus</u>: (Part I).

Let $\quad F(x) = \displaystyle\int_a^x f(t)\,dt$

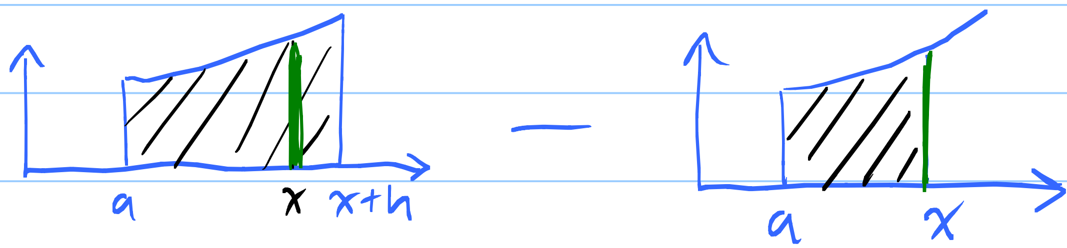
$=$ 

$y = f(t)$

Then $F'(x) = f(x)$. [F is an <u>antiderivative</u> for f.]

<u>Why it is true:</u>

$$DQ = \frac{F(x+h) - F(x)}{h} = \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$=$ 

h

$=$ 

h

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt \overset{\text{MVT } \int\text{s}}{=} \frac{1}{h} f(c)\left[\underbrace{(x+h)-x}_{h}\right]$$

$= f(c)$ for some c between x and $x+h$.

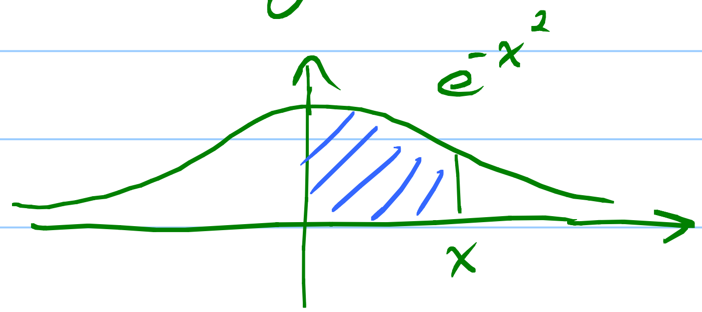
As $h \to 0$, the c's slide in to x.

$\lim_{h\to 0} DQ = f(x)$ by continuity.

So $F'(x) = f(x)$. ✓

EX: Find an antiderivative for e^{-x^2}.

$$F(x) = \int_0^x e^{-t^2}\,dt \quad \text{does it!}$$



$F(x) = \text{Area}_0^x$.

$= \text{erf}(x)$

"error function"

Fund. Thm Calc. (Part II): If $F(x)$ is an antiderivative for $f(x)$, meaning that $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Why it works: Let $G(x) = \int_a^x f(t)\,dt$.

We now have two antiderivatives for $f(x)$.

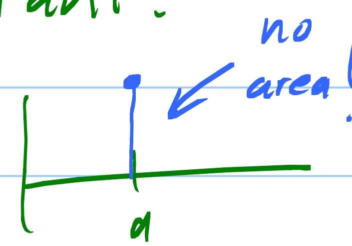
$$F'(x) = f(x)$$
$$G'(x) = f(x)$$

$$F'(x) - G'(x) = f(x) - f(x) \equiv 0.$$

$$\frac{d}{dx}\left[F(x) - G(x)\right] \equiv 0.$$

So $\boxed{F(x) - G(x) = c}$, a constant.

Notice that $G(a) = \int_a^a f(t)\,dt$



$$= 0.$$

Plug in $x = a$ to determine the c:

$$F(a) - \underbrace{G(a)}_{0} = c$$

So $c = F(a)$.

$$F(x) - G(x) = \underbrace{F(a)}_{c}.$$

Now let $x = b$:

$$F(b) - \underbrace{G(b)}_{\int_a^b f(t)\,dt} = F(a)$$

So $\int_a^b f(t)\,dt = F(b) - F(a)$. ✓