FThC, I $\qquad \dfrac{d}{dx}\left[\displaystyle\int_a^x f(t)\,dt\right] = f(x)$

II, If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

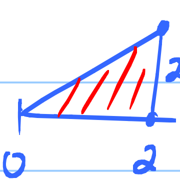
<u>EX</u>: $I = \displaystyle\int_0^{\pi} \sin x\,dx = ?$

$$\begin{cases} \dfrac{d}{dx}(\sin x) = \cos x \\ \dfrac{d}{dx}(\cos x) = -\sin x \end{cases} \qquad \begin{array}{l} f(x) = \sin x \\ \leftarrow F(x) = -\cos x \end{array}$$

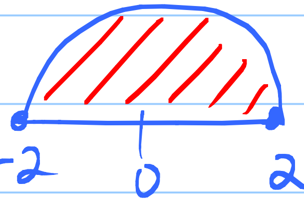
$$I = F(\pi) - F(0) = [-\cos \pi] - [-\cos 0]$$

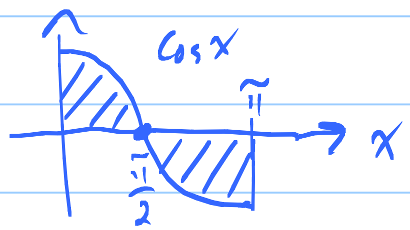
$$= [-(-1)] - [-1] = 2$$

<u>EX</u>: $\displaystyle\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \tfrac{1}{3}$

<u>Finding areas</u>: $\displaystyle\int_0^2 x\,dx =$  $= \tfrac{1}{2}\cdot 2\cdot 2 = 2$

or $= \left[\frac{1}{2}x^2\right]_0^2 = 2.$

EX: $\displaystyle\int_{-2}^{2}\sqrt{4-x^2}\,dx =$  $= \frac{1}{2}\pi\cdot 2^2.$

EX:  $\displaystyle\int_0^{\pi}\cos x\,dx = 0$

Signed area = 0

Total Area

$\displaystyle = \int_0^{\pi/2}\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\,dx$

= blue shaded total area $= 1+1 = 2$

Def^n: $\displaystyle\int_a^b f(x)\,dx = \lim_{N\to\infty}\sum_{n=1}^{N} f(x_n)\,\Delta x$

is the "Definite integral."

$\int f(x)\,dx \leftarrow$ the "Indefinite integral"

= every antiderivative for $f(x)$

$= F(x) + C$

Note: Suppose $F'(x) = f(x)$.

If $\tilde{F}'(x) = f(x)$ too, then

$$\frac{d}{dx}\left[\tilde{F}(x) - F(x)\right] = f - f \equiv 0.$$

So $\tilde{F} - F \equiv C$, a const.

$$\tilde{F} = F + C.$$

Standard integrals: $\int x^p\,dx = \frac{1}{p+1} x^{p+1} + C$

if $p \neq -1$.

If $p = -1$: $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$

$$\int e^{kx}\,dx = \frac{1}{k} e^{kx} + C.$$

EX: $x^{\pi - 1} = x^p$ with $p = \pi - 1$. (#33).

$$\pi^{x-1} = e^{\ln\left[\pi^{x-1}\right]} = e^{(x-1)\ln\pi}$$

$$= e^{(\ln\pi)x - \ln\pi} = \frac{e^{(\ln\pi)x}}{e^{\ln\pi}} = \frac{1}{\pi} e^{(\ln\pi)x}$$

$= \frac{1}{\pi} e^{kx}$ where $k = \ln\pi$.

5.5 The Chain Rule backwards.

$$\int \cos(\underbrace{2x^2+1}_{u}) \cdot \underbrace{4x}_{\frac{du}{dx}} \, dx$$

$$= \int (\cos u) \cdot \frac{du}{dx} \cdot dx$$

Think: dx's cancel.

$$= \int \frac{d}{dx}[\sin u] \, dx = \sin u + C$$

$$= \sin(2x^2+1) + C$$

Clean way to write same thing:

$$\int \cos(\underbrace{2x^2+1}_{u}) \, \underbrace{4x}_{\frac{du}{dx}} \, dx = \int \cos u \, du$$

$$= \sin u + C$$

$$= \sin(2x^2+1) + C$$

EX: $I = \int x \, (\underbrace{2x-7}_{u})^{100} \, dx$

$$\boxed{u = 2x-7}$$

$$du = \frac{du}{dx} \cdot dx$$

$$= 2 \cdot dx$$

$$du = 2\,dx$$

So $\boxed{dx = \frac{1}{2} du}$

$2x = u + 7$

$$\boxed{x = \frac{1}{2}u + \frac{7}{2}}$$

$$I = \int \left(\tfrac{1}{2}u + \tfrac{7}{2}\right) u^{100} \left(\tfrac{1}{2}\,du\right)$$

$$= \int \tfrac{1}{4}u^{101} + \tfrac{7}{4}u^{100} \; du$$

$$= \tfrac{1}{4}\int u^{101}\,du + \tfrac{7}{4}\int u^{100}\,du$$

$$= \tfrac{1}{4}\cdot\tfrac{1}{102}u^{102} + \tfrac{7}{4}\cdot\tfrac{1}{101}u^{101} + C$$

$$= \tfrac{1}{408}(2x-7)^{102} + \tfrac{7}{404}(2x-7)^{101} + C$$

EX: $I = \int \text{Tan}^3 x \; \underbrace{\text{Sec}^2 x \; dx}$

du where $u = \text{Tan}\,x$!

Let $u = \text{Tan}\,x$. Then $du = \frac{du}{dx}\cdot dx = \text{Sec}^2 x\,dx$.

$$I = \int u^3 \cdot du = \tfrac{1}{4}u^4 + C$$

$$= \tfrac{1}{4}\text{Tan}^4 x + C.$$

EX: $I = \int \frac{x}{\sqrt{\underbrace{x^2+1}_{u}}}\,dx$ Let $u = x^2+1$.

$du = 2x\cdot dx$

So $\boxed{x\,dx = \frac{1}{2}du}$

$$I = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} du = \frac{1}{2}\int \frac{1}{u^{1/2}} du = \frac{1}{2}\int u^{-1/2} du$$

$$= \frac{1}{2} \frac{1}{\left(-\frac{1}{2}+1\right)} u^{-\frac{1}{2}+1} + C$$

$$= \sqrt{u} + C = \sqrt{x^2+1} + C$$

EX: $\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \cdot dx$$

du

u

$$= \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Formulas for $\sin^2 x$ and $\cos^2 x$

$$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$$

$$\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$

$$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos \underbrace{2x}_{u} \; \underbrace{dx}_{\frac{1}{2}du}$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x)} + C$$

$$\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$