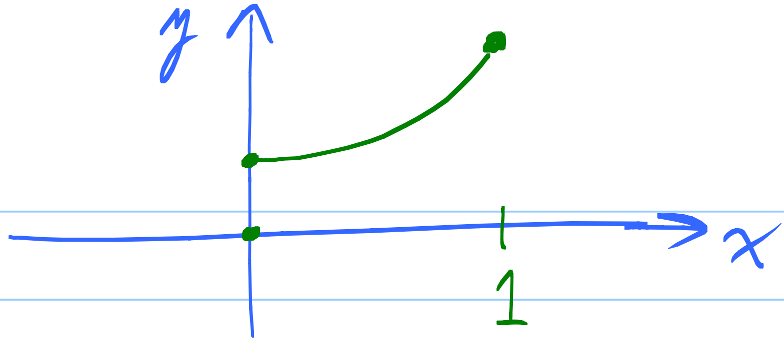
Catenary:



$$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}\left[e^{2x} - 2\underbrace{e^x e^{-x}}_{e^0 = 1} + e^{-2x}\right]$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{4}{4} + \frac{1}{4}\left[e^{2x} - 2 + e^{-2x}\right]$$

$$= \frac{1}{4}\left[e^{2x} + 2 + e^{-2x}\right]$$

$$= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 \leftarrow \text{Miracle!}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}(e^x + e^{-x})$$

$$L = \int_0^1 \frac{1}{2}(e^x + e^{-x})\, dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

Separable ODEs: $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

Initial Value Problem: $y(x_0) = y_0$.

Observe: $G(y) = F(x) + C$ defines a function y of x implicitly.

Implicit Differentiation:

$$G'(y)\frac{dy}{dx} = F'(x) + 0$$

Write $G'(y) = g(y)$ and $F'(x) = f(x)$.

solve for $\frac{dy}{dx}$: $\boxed{\frac{dy}{dx} = \frac{f(x)}{g(y)}}$

Theory of ODEs says: If we find a solution to the ODE with the given initial values, it is the unique sol^n.

Fact: $G(y) = F(x) + C$ solves $\frac{dy}{dx} = \frac{f(x)}{g(y)}$.

How to remember the formula:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

separate the variables

$$g(y)\,dy = f(x)\,dx$$

Integrate: $\displaystyle\int g(y)\,dy = \int f(x)\,dx + C$

$$\boxed{G(y) = F(x) + C}$$

Recall: $G'(y) = g(y)$ and $F'(x) = f(x)$.

EX: Solve $\boxed{\dfrac{dy}{dx} = 3x^2y^2}$ with $y(1)=5$.

$$\frac{1}{y^2}\,dy = 3x^2\,dx$$

$$\int \frac{1}{y^2}\,dy = \int 3x^2\,dx + C$$

$$-\frac{1}{y} = x^3 + C$$

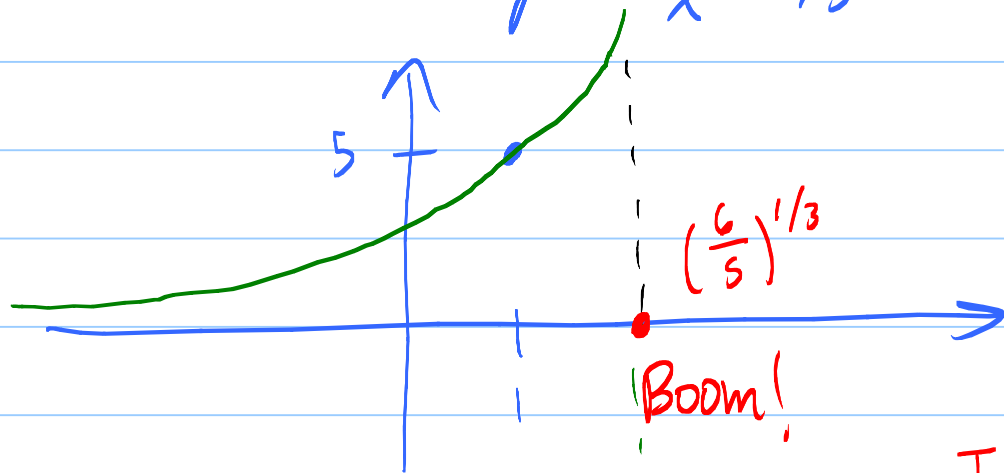
$$y = \frac{-1}{x^3+C}$$

Want $\displaystyle y(1) = \frac{-1}{1^3+C} \overset{\text{want}}{=} 5$

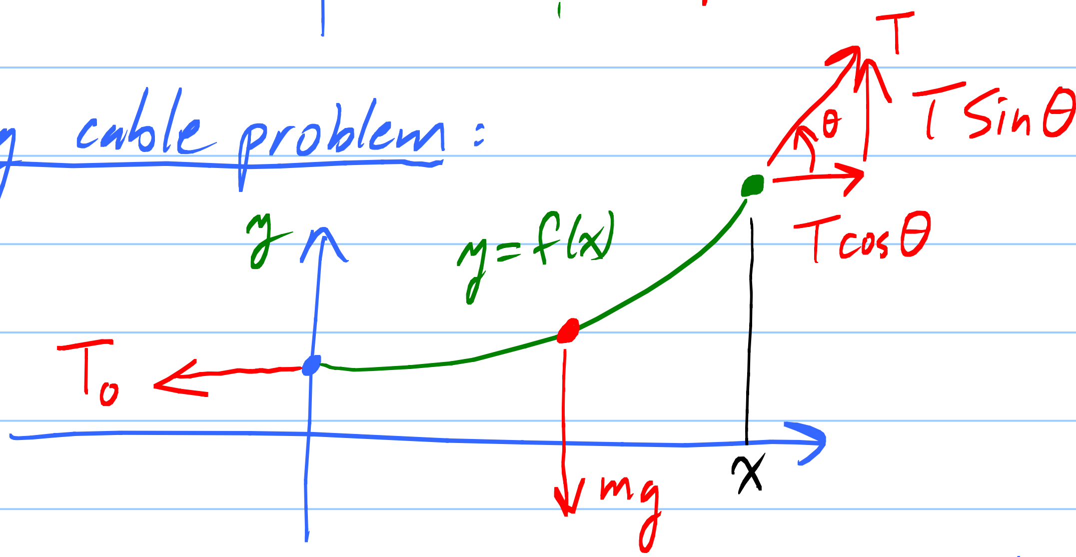
$$1+C = -1/5$$

$$\boxed{C = -6/5}$$

Unique solution: $y = \dfrac{-1}{x^3 - 6/5}$



Hanging cable problem:



Fact: Tension in rope acts in tangent direction.

$$T_0 = T\cos\theta$$

$$mg = T\sin\theta$$

Forces equalize, Motionless!

$$\frac{dy}{dx} = \text{Slope} = \tan\theta = \frac{T\sin\theta}{T\cos\theta}$$

$$= \frac{mg}{T_0}$$

But $mg = \rho L$ $\quad \rho =$ density.

Take $\rho = 1$. Then $mg = L = \displaystyle\int_0^x \sqrt{1 + f'(t)^2}\, dt$

Now $\frac{dy}{dx} = \frac{1}{T_0} \int_0^x \sqrt{1 + f'(t)^2}\, dt$

Get rid of $\int$ by differentiating and using the Fund. Thm. of Calc.

$$\frac{d^2y}{dx^2} = \frac{1}{T_0} \sqrt{1 + f'(x)^2} = \frac{1}{T_0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

2nd order ODE! Hmmm. Let $\boxed{u = \frac{dy}{dx}}$.

$$\frac{du}{dx} = \frac{d^2y}{dx^2} = \frac{1}{T_0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{T_0} \sqrt{1 + u^2}$$

Aha! Separable ODE in u and x!

$$\frac{1}{\sqrt{1+u^2}}\, du = \frac{1}{T_0}\, dx$$

$$\int \frac{1}{\sqrt{1+u^2}}\, du = \int \frac{1}{T_0}\, dx + C$$

$$\sinh^{-1} u = \frac{1}{T_0} x + C.$$

Horizontal tangent at $x = 0$. $\frac{dy}{dx} = u = 0$ when $x = 0$.

$$\underbrace{\sinh^{-1} 0}_{0} = \frac{1}{10}\cdot 0 + C$$

$$\text{so} \quad C = 0.$$

$$\sinh^{-1} u = \frac{1}{10} x$$

$$u = \sinh\left(\frac{x}{10}\right)$$

$$\uparrow \frac{dy}{dx}$$

$$y = \int \sinh\left(\frac{x}{10}\right) dx = 10 \cosh\left(\frac{x}{10}\right) + C$$

What is $\frac{d}{dx} \sinh^{-1} x$?

$$y = \sinh^{-1} x$$

$$x = \sinh y \leftarrow$$

Differentiate: $\underbrace{\frac{dx}{dx}}_{1} = \underbrace{\frac{d}{dy}\left[\sinh y\right]}_{\cosh y} \cdot \frac{dy}{dx}$

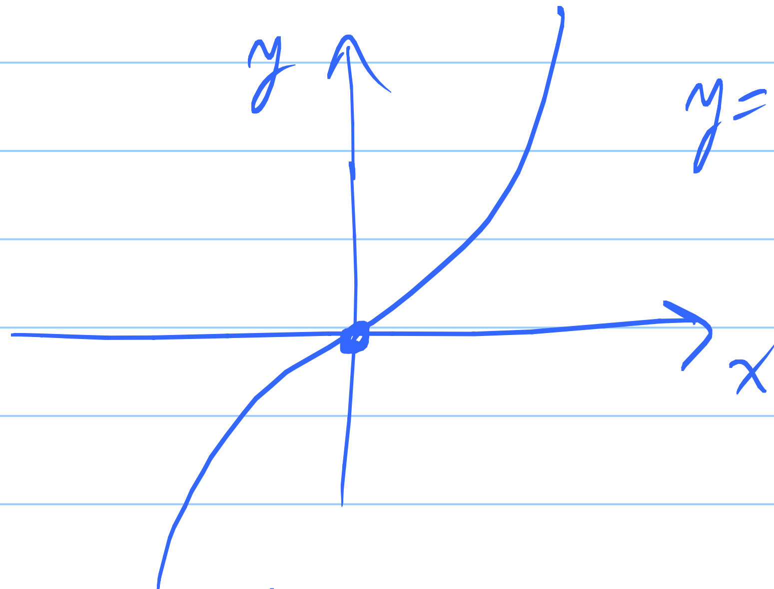
$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$$\cosh^2 y - \sinh^2 y = \left[\tfrac{1}{2}(e^y + e^{-y})\right]^2 - \left[\tfrac{1}{2}(e^y - e^{-y})\right]^2$$

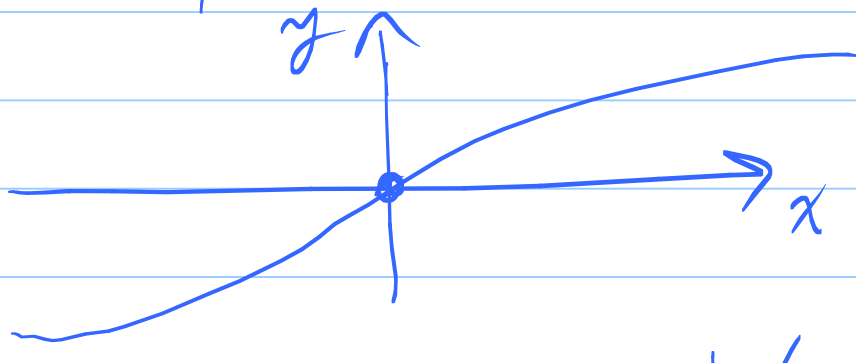
$$= \underline{1}$$

$$\cosh y = \sqrt{1 + \sinh^2 y}$$

So $$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + C$$



$$y = \sinh x = \tfrac{1}{2}(e^x - e^{-x})$$



$$y = \sinh^{-1} x$$

Trig functions: $\begin{cases} \sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta}) \\ \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \end{cases}$

$$y'' + y = 0$$

Hyperbolic functions: $y'' - y = 0$