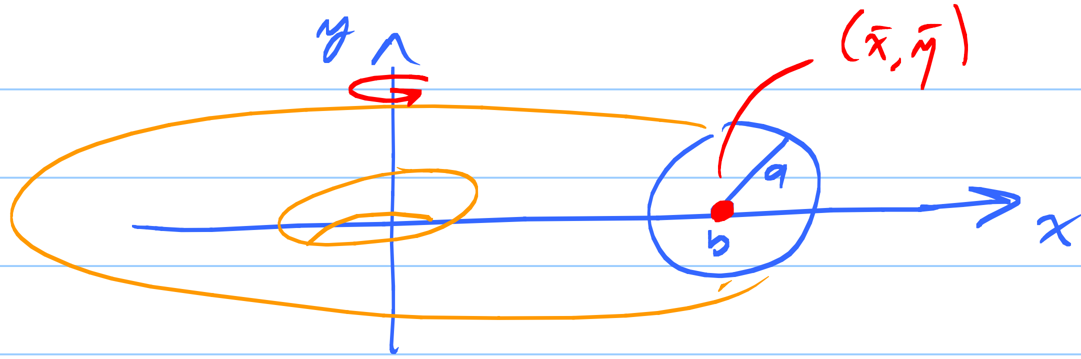
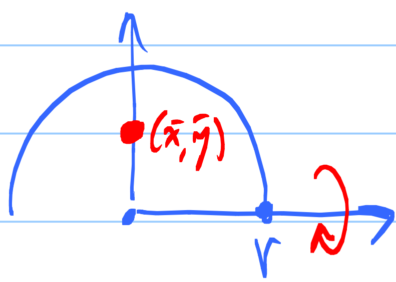
EX: Area of a bagel.



2nd Thm Pappus: $S = (2\pi \bar{x})\, L$

$$= (2\pi b)(2\pi a) = 4\pi^2 ab$$

Prob: Find the centroid of a semi-circle wire.



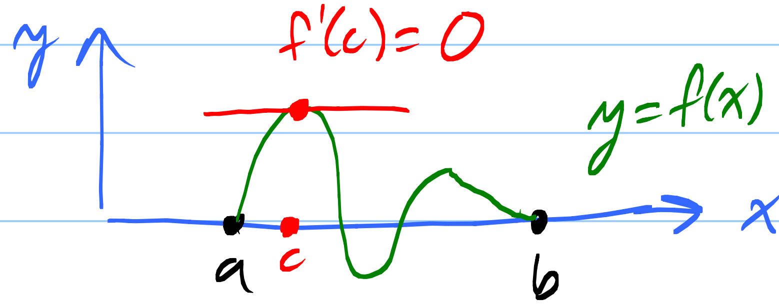
Sphere Area $= (2\pi \bar{y})\, L$

$$4\pi r^2 = (2\pi \bar{y})(\pi r)$$

$$\bar{y} = \frac{2r}{\pi}$$

Mean Value Theorem I:

Rolle's Theorem:



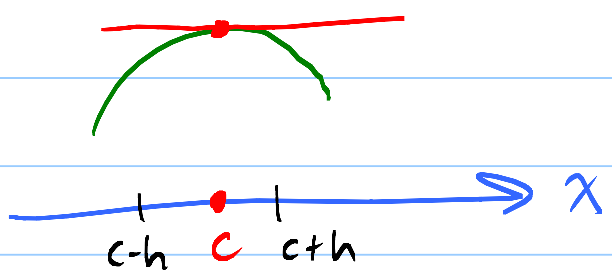
Hypotheses: f continuous on $[a,b]$.
f diff'ble on (a,b).

How to show that the point c exists:

Fact: A continuous function on a closed interval $[a,b]$ assumes its maximum value M at a point in the interval. [To prove this, need to know that real numbers $\mathbb{R}$ are "complete." The rational numbers $\mathbb{Q}$ are not complete. Let $S = \{x \in \mathbb{Q} : x^2 \leq 2\}$. Take $x_n \in S$ with $x_n^2 \to 2$. Can show $x_n \to \sqrt{2}$. $\sqrt{2}$ is not rational. $\mathbb{Q}$ has holes. $\mathbb{R}$ does not.]

Assume the Fact. Then there is a c in $[a,b]$ where $f(c) = M$, $M = $ max of f on $[a,b]$.

Claim: $f'(c)$ must be zero.



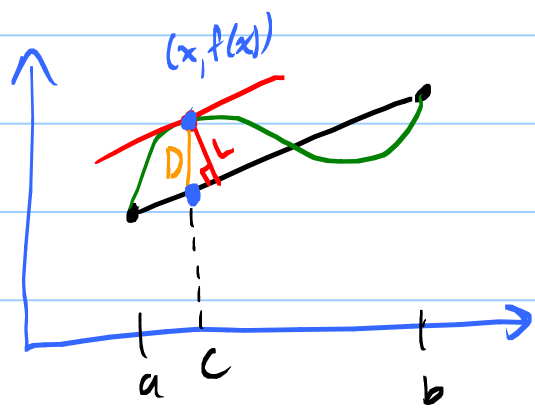
Max

$$DQ_+ = \frac{f(c+h) - f(c)}{h} \leq 0$$

$$DQ_- = \frac{f(c-h) - f(c)}{-h} \geq 0$$

$$0 \geq \lim_{h \to 0} DQ_+ = \underbrace{f'(c)}_{\text{must } = 0.} = \lim_{h \to 0} DQ_- \geq 0$$

How to cook up the c in MVT:



D is maximized exactly where L is.

$$D(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$$

Notice that $D(a)=0$, $D(b)=0$. Rolle's gives a c with $D'(c)=0$. But

$$D'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

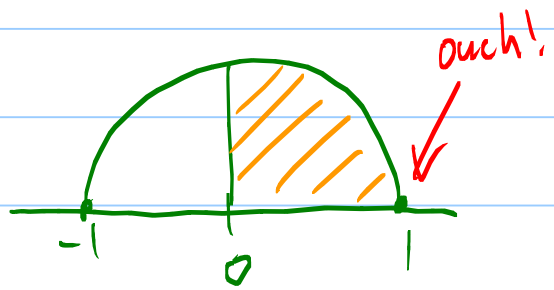
$$0 = D'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

Consequences:

1) Rectangle Rule estimate

$$|I - RS| \le \frac{M(b-a)^2}{N}, \quad M = \max_{[a,b]} |f'|,$$

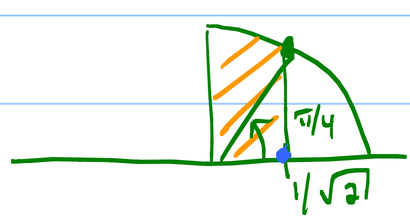
Prob: Compute π by Rec. Rule to 6 decimal places.



$$\frac{\pi}{4} \approx \sum_{n=1}^{N} \sqrt{1-\left(\frac{n}{N}\right)^2} \cdot \frac{1}{N}$$

$|f'(x)| \to \infty$ as $x \to 1$. $M = \infty$.

Better



$\left(\text{Function involving } \pi\right) \approx RS.$

$$\text{Error} \le \frac{M\left(1/\sqrt{2}-0\right)^2}{N} \le \frac{1/2}{N}$$

Big Fact: If $f'(x) \equiv 0$, then f is constant.

Why: $\dfrac{f(b)-f(a)}{b-a} \overset{MVT}{=} \underbrace{f'(c)}_{0}$. So $f(b) = f(a)$

Hold a still and let b slide. See $f(x) \equiv f(a)$.

Big Fact 2: If $f'(x) \equiv g'(x)$, then $f(x) = g(x) + C$ where C is a constant.

EXAMPLE: $f(x) = \ln(Kx) = \ln u$ where $u = kx$

$$f'(x) = \frac{1}{u}\frac{du}{dx} = \frac{1}{Kx}(K) = \frac{1}{x}$$

Hmmm. $g(x) = \ln x$ has the same derivative.

So $\ln(Kx) = \ln x + C$.

Plug $x = 1$: $\ln K = \underbrace{\ln 1}_{0} + C$. So $C = \ln K$.

We have shown: $\ln(Kx) = \ln K + \ln x$.

Similary, can show $\ln(x^r) = r \ln x$.

L'Hopital's Rule $\dfrac{f(x)}{g(x)}$ $\quad f(a) = 0$, $g(a) = 0$ $\quad \dfrac{0}{0}$ at a.

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overset{0}{\overset{\shortparallel}{f(a)}}}{g(x) - \underset{\underset{0}{\shortparallel}}{g(a)}} = \frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)} = \frac{f'(c_1)}{g'(c_2)}$$

as $x \to a$, the c's slide into a.

Limit is $\dfrac{f'(a)}{g'(a)}$!