L'Hôpital's Rule

$f(a) = 0$
$g(a) = 0$

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overset{0}{f(a)}}{g(x) - \underset{0}{g(a)}} \cdot \frac{\left(\frac{1}{x-a}\right)}{\left(\frac{1}{x-a}\right)} = \frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)}$$

$$= \frac{f'(c_1)}{g'(c_2)}$$

As $x \to a$, both c's get squeezed down to a.

$$\longrightarrow \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

← need $g'(a) \neq 0$ to do this here.

Fancier version: Suppose f and g are diff'ble on $(a-h, a+h)$ and $f(a) = g(a) = 0$. Suppose further that $g'(x) \neq 0$ if $x \neq a$ on $(a-h, a+h)$. Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the right hand limit makes sense.

Fancier fancier version: Can put $\lim_{x \to a+}$ or $\lim_{x \to a-}$ in place $\lim_{x \to a}$. Can have indeterminate form $\frac{\infty}{\infty}$ in place of $\frac{0}{0}$.

Can even let $a = +\infty$ or $-\infty$.

EX: $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} \overset{0/0}{=} \lim\limits_{x \to a} \dfrac{f'(x)}{1} = f'(a)$

Warning! L'H only applies if you have indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$1 = \lim\limits_{x \to 0} \dfrac{\cos x}{1 + x^2} \neq \lim\limits_{x \to 0} \dfrac{-\sin x}{2x} = \dfrac{-1}{2}$

EX: $\lim\limits_{x \to 0} \dfrac{\sin x}{x} \overset{0/0}{=} \lim\limits_{x \to 0} \dfrac{\cos x}{1} = \cos 0 = 1$

EX: $\lim\limits_{x \to \infty} \dfrac{x^3}{e^x} \overset{\infty/\infty}{=} \lim\limits_{x \to \infty} \dfrac{3x^2}{e^x} \overset{\infty/\infty}{=} \lim\limits_{x \to \infty} \dfrac{6x}{e^x}$

$\overset{\infty/\infty}{=} \lim\limits_{x \to \infty} \dfrac{6}{e^x} = 0$

"e^x grows faster than any power of x!"

EX: $\lim\limits_{x \to \infty} \dfrac{x^p}{\ln x} \overset{\infty/\infty}{=} \lim\limits_{x \to \infty} \dfrac{px^{p-1}}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} px^p$

$(p > 0)$

$= \infty$

"$\ln x$ grows slower than $x^{1/1{,}000}$!"

Prob: $\lim\limits_{x \to 0} \left(\underbrace{\frac{1}{\sin x} - \frac{1}{x}}_{= \frac{x - \sin x}{x \sin x}} \right)$

$x \to 0+$: $\infty - \infty$

$x \to 0-$: $-\infty - (-\infty)$

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \overset{\frac{0}{0}}{=}$$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x - x \sin x + \cos x} = \frac{0}{2} = 0$$

EX: $\lim\limits_{x \to \infty} (x^3 - x^2) \overset{\infty - \infty}{=} \infty$

$$x^3 - x^2 = x^2 (x - 1) \to \infty$$

How big is a big set?

Al	Sue	Jim	. . . .	Sam
$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$
1	2	3		30

$\{1, 2, 3, \ldots, 30\}$ one-to-one correspondence with people in room.

Georg Cantor.

There are just as many even numbers as numbers.

$$\begin{array}{cccccc} 2 & 4 & 6 & 8 & 10 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\ 1 & 2 & 3 & 4 & 5 & \end{array}$$

one-to-one map is $n \mapsto 2n$

There are just as many rational numbers as there are natural numbers.

....

$10\ \frac{1}{6}$ $\quad \cancel{\frac{2}{6}}$ $\quad \cancel{\frac{3}{6}}$ $\quad \cancel{\frac{4}{6}}$ $\quad 11\ \frac{5}{6}$

$6\ \frac{1}{5}$ $\quad 7\ \frac{2}{5}$ $\quad 8\ \frac{3}{5}$ $\quad 9\ \frac{4}{5}$

$4\ \frac{1}{4}$ $\quad \cancel{\frac{2}{4}}$ $\quad 5\ \frac{3}{4}$

$2\ \frac{1}{3}$ $\quad 3\ \frac{2}{3}$

$1\ \frac{1}{2}$

Cantor showed that $\mathbb{R}$ is bigger than $\mathbb{N}$.

$1 \longleftrightarrow .3479\ldots$

$2 \longleftrightarrow .6213\ldots$

$3 \longleftrightarrow .7892\ldots$

$4 \longleftrightarrow .9318\ldots$

$\vdots$

$$.\underset{\text{not } 3}{4}\ \underset{\text{not } 2}{5}\ \underset{\text{not } 9}{7}\ \underset{\text{not } 8}{6}\ \ldots$$

Ouch! a number that doesn't get hit.

$\mathbb{R}$ is "un-countable."

There is no biggest set.

Given set S, the "power set," which is the set of all subsets of S is always bigger.

Book: Ralph P. Boas and Harold P. Boas, "A primer of real functions." p. 8-20.