

L'Hôpital's Rule

$$f(a) = 0$$

$$g(a) = 0$$

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overset{0}{f(a)}}{g(x) - \underset{0}{g(a)}} \cdot \frac{\left(\frac{1}{x-a}\right)}{\left(\frac{1}{x-a}\right)} = \frac{\left(\frac{f(x) - f(a)}{x-a}\right)}{\left(\frac{g(x) - g(a)}{x-a}\right)}$$

$$= \frac{f'(c_1)}{g'(c_2)}$$

As $x \rightarrow a$, both c 's get squeezed down to a .

$$\rightarrow \frac{f'(a)}{g'(a)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

need $g'(a) \neq 0$ to do this here.

Fancier version: Suppose f and g are diff'ble on $(a-h, a+h)$ and $f(a) = g(a) = 0$. Suppose further that $g'(x) \neq 0$ if $x \neq a$ on $(a-h, a+h)$.

$$\text{Then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \text{ if the}$$

right hand limit makes sense.

Fancier fancier version: Can put $\lim_{x \rightarrow a^+}$

or $\lim_{x \rightarrow a^-}$ in place of $\lim_{x \rightarrow a}$. Can have

indeterminate form $\frac{\infty}{\infty}$ in place of $\frac{0}{0}$.

Can even let $a = +\infty$ or $-\infty$.

$$\underline{\text{EX:}} \quad \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \stackrel{0/0}{=} \lim_{x \rightarrow a} \frac{f'(x)}{1} = f'(a)$$

Warning! L'H only applies if you have indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$1 = \lim_{x \rightarrow 0} \frac{\cos x}{1+x^2} \neq \lim_{x \rightarrow 0} \frac{-\sin x}{2x} = -\frac{1}{2}$$

$$\underline{\text{EX:}} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{0/0}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$\underline{\text{EX:}} \quad \lim_{x \rightarrow \infty} \frac{x^3}{e^x} \stackrel{\infty/\infty}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \stackrel{\infty/\infty}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x}$$

$$\stackrel{\infty/\infty}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0$$

" e^x grows faster than any power of x !"

$$\underline{\text{EX:}} \quad \lim_{x \rightarrow \infty} \frac{x^p}{\ln x} \stackrel{\infty/\infty}{=} \lim_{x \rightarrow \infty} \frac{px^{p-1}}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} px^p = \infty$$

$(p > 0)$

" $\ln x$ grows slower than $x^{1/1,000}$!" 3

Prob: $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

$x \rightarrow 0^+$: $\infty - \infty$ $= \frac{x - \sin x}{x \sin x}$

$x \rightarrow 0^-$: $-\infty - (-\infty)$

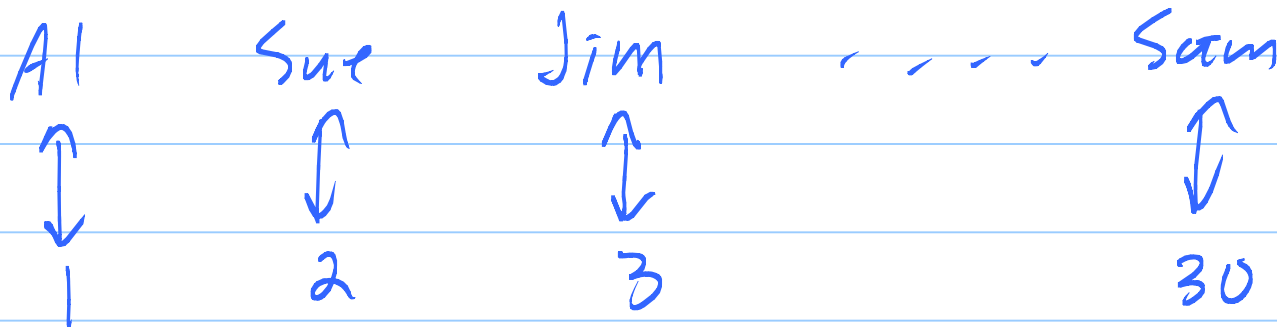
$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} = \frac{0}{0}$

$= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x - x \sin x + \cos x} = \frac{0}{2} = 0$

EX: $\lim_{x \rightarrow \infty} (x^3 - x^2) = \infty$

$x^3 - x^2 = x^2(x-1) \rightarrow \infty$

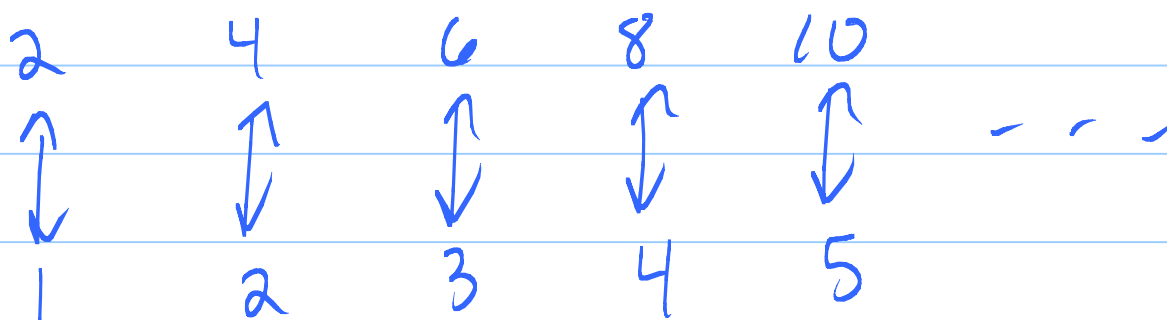
How big is a big set?



$\{1, 2, 3, \dots, 30\}$ one-to-one correspondence
with people in room.

Georg Cantor.

There are just as many even numbers as numbers.



one-to-one map is $n \mapsto 2n$

There are just as many rational numbers as
there are natural numbers.

10 $\frac{1}{6}$ ~~$\frac{2}{6}$~~ ~~$\frac{3}{6}$~~ ~~$\frac{4}{6}$~~ 11 $\frac{5}{6}$

6 $\frac{1}{5}$ 7 $\frac{2}{5}$ 8 $\frac{3}{5}$ 9 $\frac{4}{5}$

4 $\frac{1}{4}$ ~~$\frac{2}{4}$~~ 5 $\frac{3}{4}$

2 $\frac{1}{3}$ 3 $\frac{2}{3}$

1 $\frac{1}{2}$

Cantor showed that $\mathbb{R}$ is bigger than $\mathbb{N}$. 5

1	$\longleftrightarrow$	.	3	4	7	9	...
2	$\longleftrightarrow$	.	6	2	1	3	...
3	$\longleftrightarrow$	.	7	8	9	2	...
4	$\longleftrightarrow$	.	9	3	1	8	...
		:					

	4	5	7	6	...
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	
	not 3	not 2	not 9	not 8	

Ouch!
a number
that doesn't
get hit.

$\mathbb{R}$ is "un-countable."

There is no biggest set.

Given set S , the "power set," which is the set of all subsets of S is always bigger.

Book: Ralph P. Boas and Harold P. Boas,
"A primer of real functions." p. 8-20.