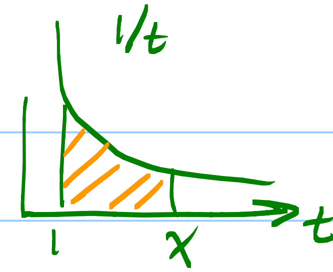
Exam 1 Tues. Oct. 4 in class.

HWK 6 due Thurs. Oct. 6

7.1 **Define** $\ln x = \int_1^x \frac{1}{t}\,dt =$ 



$\ln 1 =$ "no area" $= 0$

If $0 < x < 1$, then $\ln x = \int_1^x \frac{1}{t}\,dx = -\int_x^1 \frac{1}{t}\,dx$

$= -$(area from $x$ to $1$ under $\frac{1}{t}$).

**Fact 1**: Fund. Thm. Calc.:

$$\frac{d}{dx}\ln x = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

**Fact 2**: $\ln(ab) = \ln a + \ln b$.

$\frac{d}{dx}\ln(ax) = \frac{1}{(ax)}\cdot a = \frac{1}{x} \Leftarrow$ same as $\frac{d}{dx}\ln x$

So $\ln(ax) = \ln x + C$. Plug $x=1$ to see $C = \ln a$.

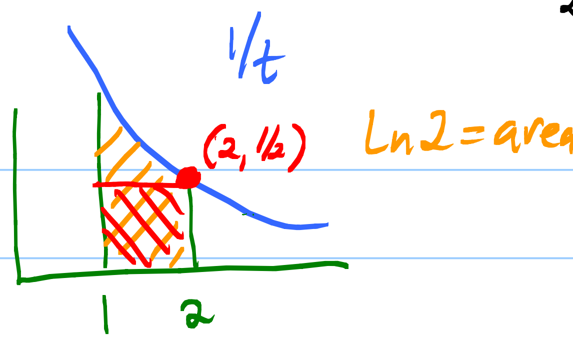
**Fact 3**: Similarly, $\ln x^r = r \ln x$.

**Other laws**: $\ln \frac{1}{x} = \ln x^{-1} = -\ln x$.

**Fact 4**: $\ln x$ is increasing.

Why: $\frac{d}{dx}\ln x = \frac{1}{x} > 0$ for $x > 0$.

<u>Fact 5</u>: $\lim_{x\to\infty} \ln x = \infty$.




$$\ln 2^n = n \underbrace{\ln 2}_{> 1\cdot\frac{1}{2}}$$

So $\ln 2^n > \frac{n}{2}$. This plus $\ln x$ <u>increasing</u> shows that $\lim_{x\to\infty} \ln x = \infty$.

<u>Fact 6</u>: $\lim_{x\to 0+} \ln x = -\infty$.

$$\ln 2^{-n} = -n \ln 2 < -\frac{n}{2}.$$

Conclude: $\ln x$ maps $(0,\infty)$ one-to-one onto $(-\infty,\infty)$.

<u>Next</u>: <u>Define</u> $\exp(x) = f^{-1}(x)$ where $f(x) = \ln x$.

$\exp(x)$ is defined for <u>all</u> $x \in \mathbb{R}$.

<u>Facts</u>: $\exp(\ln x) = x$ for $x > 0$

$\ln \exp(x) = x$ for all $x$.

$\exp(x) > 0$ for all $x$.

$\exp(0) = \exp(\ln 1) = 1$.

Basic rules: $\exp(a+b) \stackrel{?}{=} \exp(a)\exp(b)$.

True equality if and only if the Ln of both sides are the same (because Ln is 1-1).

$$\operatorname{Ln}(\exp(a+b)) \stackrel{?}{=} \operatorname{Ln}\left[\exp(a)\exp(b)\right]$$

$$a+b \stackrel{?}{=} \underbrace{\operatorname{Ln}(\exp(a))}_{a} + \underbrace{\operatorname{Ln}(\exp(b))}_{b}$$

yes!

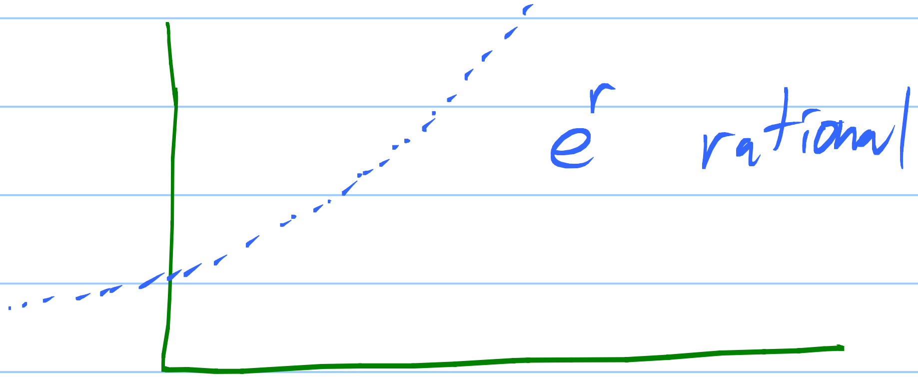
Rest of rules follow similarly.

$e = \exp(1) = f^{-1}(1) =$ "the $x$ such that $\int_1^x \frac{1}{t}\,dt = 1$"

Book shows that $e^r = \exp(r)$ when $r$ is rational.

Define $e^{\pi} = \exp(\pi)$.

$e^{p/q} = (e^p)^{1/q}$




Note: $\frac{d}{dx}\operatorname{Ln}|x| = \frac{1}{x}$ when $x<0$.

So $\int \frac{1}{x}\,dx = \operatorname{Ln}|x| + C$ works for $x>0$ ✓ $x<0$ too.

$$\int \tan x \, dx = \int \frac{\sin x}{\underbrace{\cos x}_{u}} dx \quad (\sin x \, dx = -du)$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

---

a: look up $\log_{10} a$

b: look up $\log_{10} b$

$\underbrace{\log_{10} a + \log_{10} b}_{\log_{10} ab}$ $\leftarrow$ look up in reverse. Get ab.

---

7.2 Seperable ODEs.

$$\frac{dy}{dx} = e^{2x-3y} = e^{2x} e^{-3y} = \frac{e^{2x}}{e^{3y}}, \quad y(0) = 0$$

$$e^{3y} dy = e^{2x} dx$$

$$\int e^{3y} dy = \int e^{2x} dx$$

$$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2x} + C$$

Solve for $C$ now!

$$\frac{1}{3}e^{3\cdot 0} = \frac{1}{2}e^{2\cdot 0} + C$$

$$\frac{1}{3} = \frac{1}{2} + C \qquad \boxed{C = -\frac{1}{6}}$$

$$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2x} - \frac{1}{6}.$$

$$e^{3y} = \frac{3}{2}e^{2x} - \frac{1}{2}$$

$$3y = \ln(e^{3y}) = \ln\left[\frac{3}{2}e^{2x} - \frac{1}{2}\right]$$

$$\boxed{y = \frac{1}{3}\ln\left(\frac{3}{2}e^{2x} - \frac{1}{2}\right)}$$

## Famous separable ODEs

1) Population growth: $\dfrac{dP}{dt} = kP$

$$P = P_0 e^{kt} \quad \text{where} \quad P_0 = P(0).$$

2) Radioactive decay: $\dfrac{dR}{dt} = -kR$

$$R = R_0 e^{-kt} \quad \text{where} \quad R_0 = R(0).$$