This week only: Recitation on Thursday with C. He.
Lecture on Friday with me.

p. 434: 8. Just plug $y = \frac{x}{\ln x}$ into ODE to check it really is a solution to the ODE satisfying the initial conditions.

$y = \sinh^{-1} x$ $\qquad$ $x = \sinh y$ ← differentiate, use chain rule

$$1 = \frac{dx}{dx} = (\cosh y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\cosh(\sinh^{-1} x)}$$ ← true, but ugh!

Better: $\cosh^2 y - \sinh^2 y = 1$

$$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1+x^2}$$

$$\frac{1}{\sqrt{1+x^2}}$$

$$\boxed{\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}}$$

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \sinh^{-1} x + C$$

What is $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$? $a>0$

$$= \int \frac{1}{a\sqrt{1+\left(\frac{x}{a}\right)^2}}\,dx \qquad u=\frac{x}{a},\ du=\frac{1}{a}dx$$

$$= \int \frac{1}{\sqrt{1+\underset{u}{\left(\frac{x}{a}\right)}^2}} \underbrace{\left(\frac{1}{a}\,dx\right)}_{du} = \int \frac{1}{\sqrt{1+u^2}}\,du$$

$$= \sinh^{-1} u + C$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + C$$

What is $\int \frac{1}{\sqrt{3+4x^2}}\,dx$?

$a=\sqrt{3}$ $u=2x$

$u=2x$

$du=2dx$

$dx=\frac{1}{2}du$

$$\int \frac{1}{\sqrt{a^2+u^2}}\left(\frac{1}{2}du\right) = \frac{1}{2}\sinh^{-1}\frac{u}{a} + C$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

Fun thing:

$$y = \sinh^{-1} x$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{e^y - \frac{1}{e^y}}{2}$$

Multiply by e^y: $2x\,e^y = (e^y)^2 - 1$

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Quadratic in e^y!

$$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-2x) \pm \sqrt{(-2x)^2-(-4)}}{2\cdot 1}$$

always >0

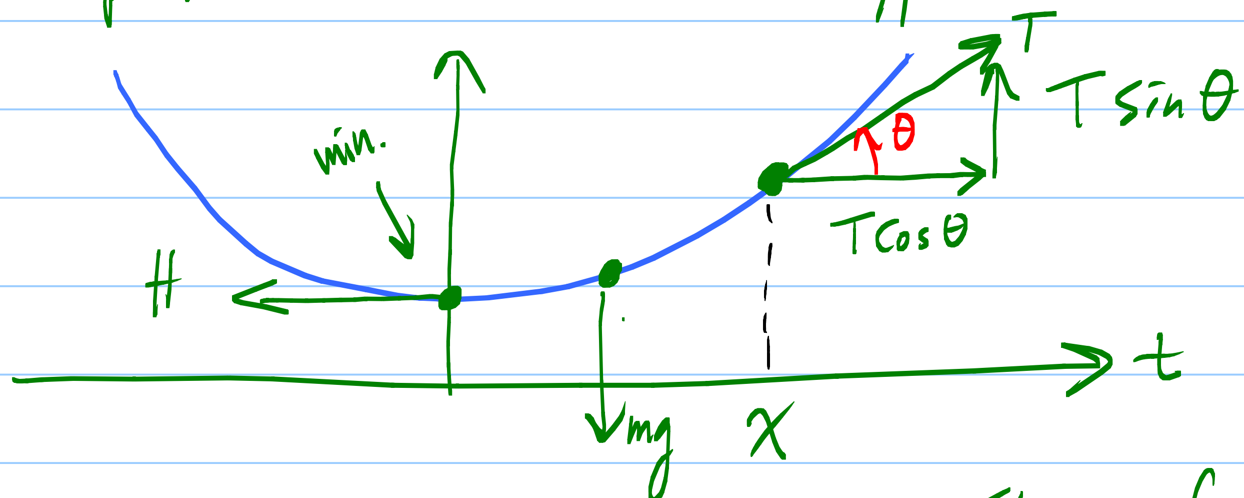
$$= x \pm \sqrt{x^2+1}$$

the minus makes this negative.

$$e^y = x + \sqrt{x^2+1}$$

$$y = \ln\left(x+\sqrt{x^2+1}\right) = \sinh^{-1} x \;!$$

See p. 442 for other inverse hyperbolic functions.



Shape of cable: $y = f(x)$.

Statics: $H = T\cos\theta$

$$mg = T\sin\theta$$

$$\rho\int_0^x \sqrt{1+f'(t)^2}\,dt = T\sin\theta$$

$$f'(x) = \text{slope at } x = \tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{\rho\int_0^x \sqrt{1+f'(t)^2}\,dt}{H}$$

$$f'(x) = \frac{\rho}{H}\int_0^x \sqrt{1+f'(t)^2}\,dt$$

$$f''(x) = \frac{\rho}{H}\sqrt{1+f'(x)^2}$$ ← Fund Thm Calc.

Now let $u = f'(x)$.

$$\frac{du}{dx} = \frac{\rho}{H}\sqrt{1+u^2}$$ ← Separable ODE

$$\frac{du}{\sqrt{1+u^2}} = \frac{\rho}{H}\,dx$$

$$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\rho}{H}\,dx$$

Note: $f'(0)=0$
$u = f'(x)$.

$$\sinh^{-1} u = \frac{\rho}{H}x + C.$$

$$\underbrace{\sinh^{-1} 0}_{0} = \frac{\rho}{H}\cdot 0 + C$$

$$\boxed{C=0}$$

$$\sinh^{-1} u = \frac{\rho}{H}x$$

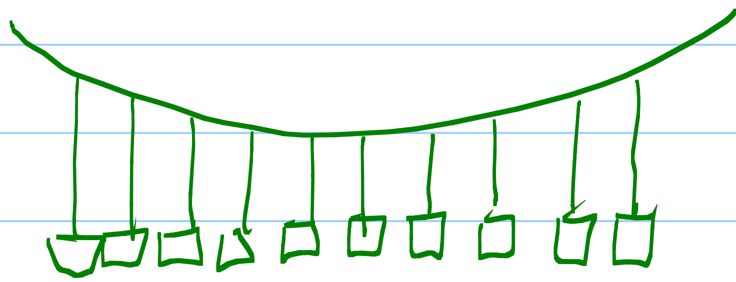
$$\underbrace{u}_{f'(x)} = \sinh\left(\frac{\rho}{H}x\right)$$

Finally, $f(x) = \int \sinh\left(\frac{\rho}{H}x\right)dx$

$$= \frac{H}{\rho}\cosh\left(\frac{\rho}{H}x\right) + C$$

See 7.3: 83, 84 for more.

Bridges:



Parabola

7.4 Relative rates of growth

"e^x grows faster than x^{100} as $x \to \infty$"

meaning $\lim\limits_{x\to\infty} \dfrac{e^x}{x^{100}} = \infty$.

"$\sqrt{x^2+1}$ and x grow at the same rate as $x \to \infty$"

$$\frac{\sqrt{x^2+1}}{x} = \sqrt{1+\frac{1}{x^2}} \to \sqrt{1+0} = 1 \quad \text{as } x\to\infty.$$

We say $f(x)$ and $g(x)$ grow at same rate as $x \to \infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, \quad L > 0.$$

"$\ln x$ grows slower than $x^{1/1000}$ as $x \to \infty$"
means $\lim_{x\to\infty} \frac{\ln x}{x^{1/1000}} = 0$.

Read about "little-oh" and "big-oh" notation p. 446-447.

<u>Prob</u>: Compute $\lim_{n\to\infty} \Big(\underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}}_{S_n} \Big)$

$$S_n = \frac{1/n}{1+1/n} + \frac{1/n}{1+\frac{2}{n}} + \cdots\cdots + \frac{1/n}{1+1}$$

Aha! Think $\Delta x = \frac{1}{n}$, $f(x) = \frac{1}{x}$.

$$S_n = \text{Riemann Sum} = \sum_{k=1}^{n} f\left(1+\frac{k}{n}\right)\Delta x$$

$$\to \int_1^2 \frac{1}{x}\,dx = \ln 2 - \underbrace{\ln 1}_{0} = \ln 2.$$

Similar problem: $\frac{1^5 + 2^5 + \cdots + n^5}{n^6}$

$$= \left(\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5 \right) \cdot \frac{1}{n}$$

Δx

$$\to \int_0^1 x^5\,dx = \frac{1}{6}$$