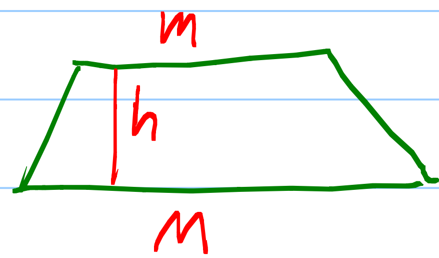
Numerical Methods

Rectangle Rule. $I = \int_a^b f(x)\,dx$

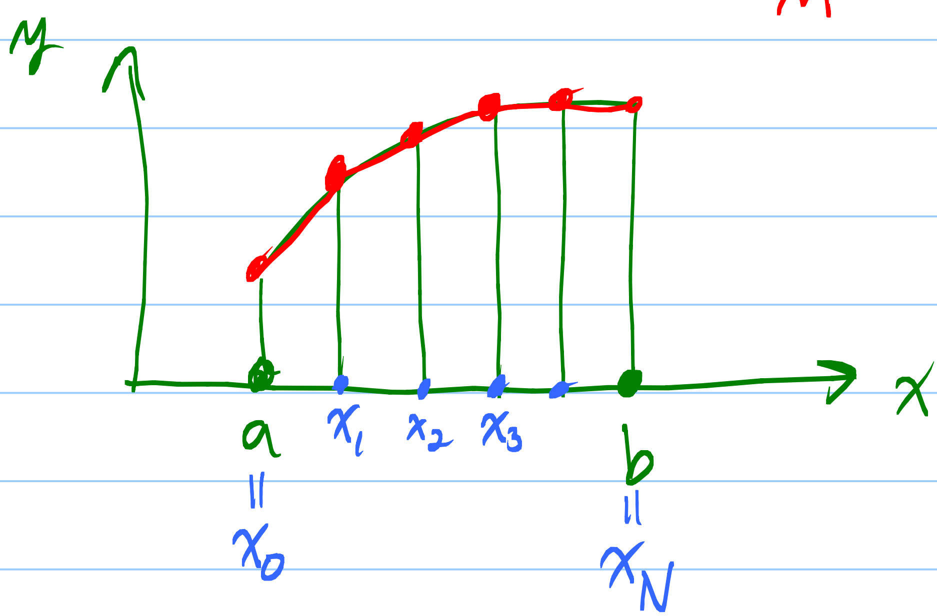
$$RS = \sum_{n=1}^{N} f(a + n\Delta x)\,\Delta x \qquad \Delta x = \frac{b-a}{N}$$

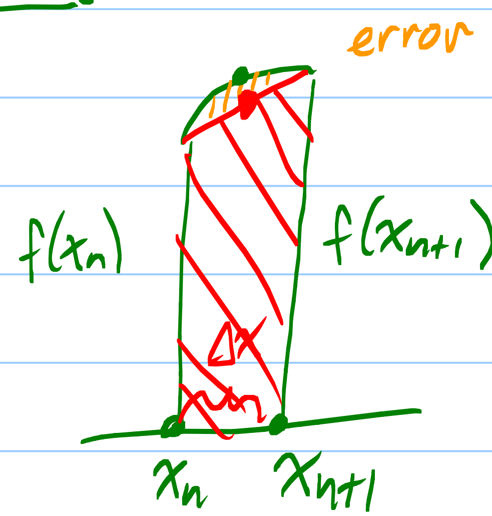
$$|I - RS| < \frac{M(b-a)^2}{N} \quad \text{where } M = \max_{[a,b]} |f'|.$$

Trapezoid Rule:



$$A = \frac{m+M}{2} \cdot h$$





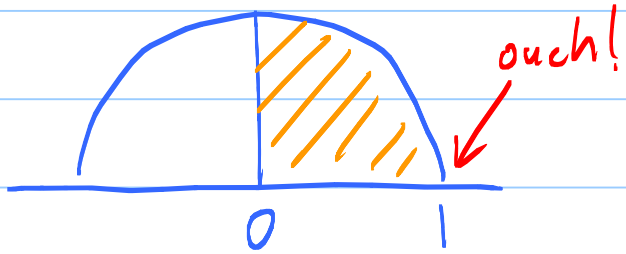
$$\text{Area} = \frac{f(x_n) + f(x_{n+1})}{2} \cdot \Delta x$$

$$TS = \left[\frac{f(a)}{2} + \sum_{n=1}^{N-1} f(x_n) + \frac{f(b)}{2}\right] \Delta x$$

Much better than Rect. Rule!

$$|I - TS| < \frac{\mathcal{M}(b-a)^3}{12N^2}, \quad \mathcal{M} = \max_{[a,b]} |f''|$$

Problem: Compute π.

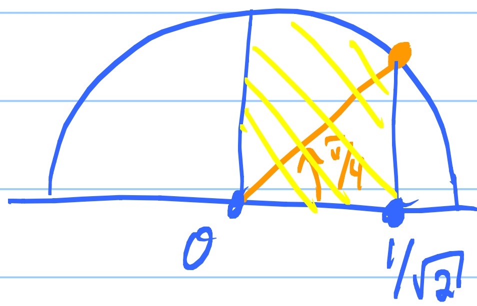


$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

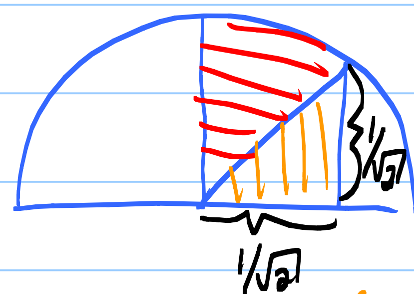
$$M = \max_{[0,1]} |f'| = \infty \,!$$

Rect. Rule iffy.

Better:



$A = \frac{\pi}{8}$

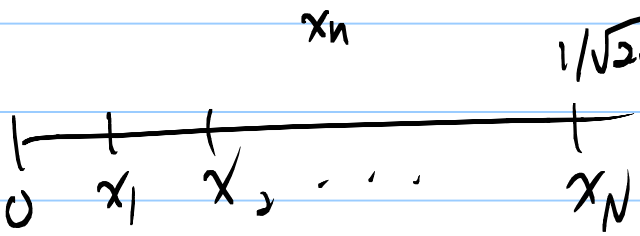


$A = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1/4$

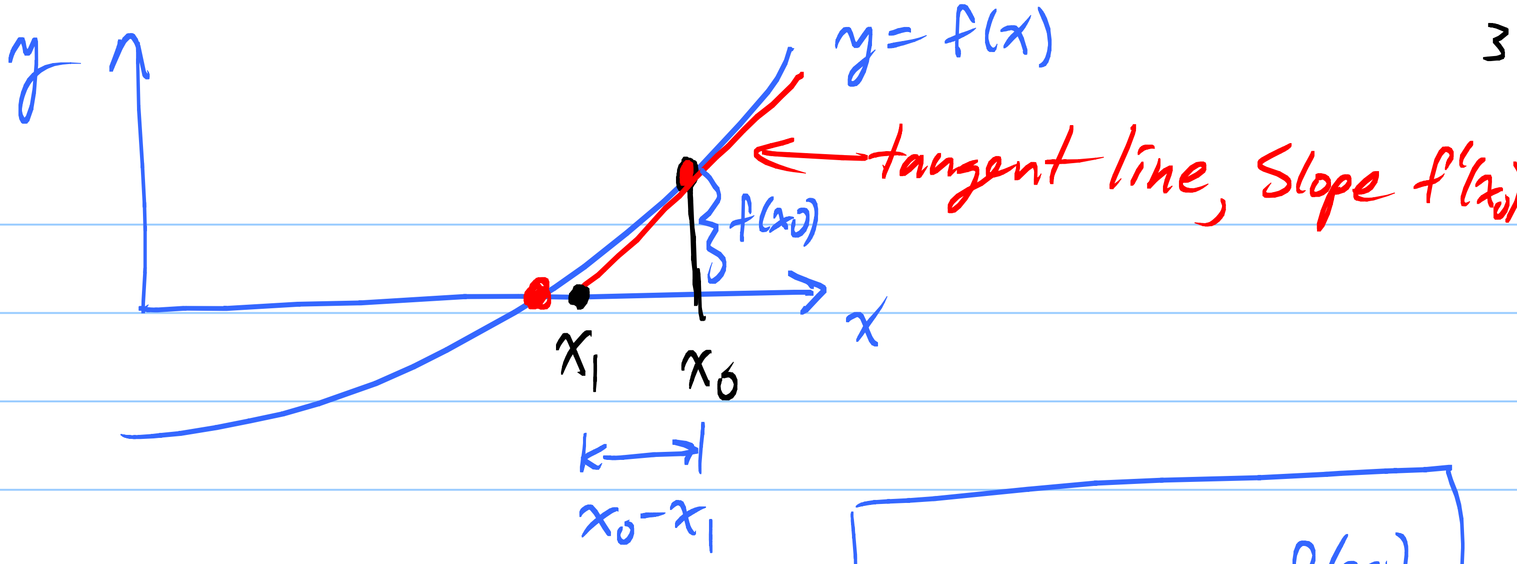
So $$\underbrace{\int_0^{1/\sqrt{2}} \sqrt{1-x^2}\,dx}_{RS = \sum_{n=1}^{N} \sqrt{1-\left(\frac{n}{\sqrt{2}N}\right)^2}\,\cdot\,\Delta x} = \frac{\pi}{8} + \frac{1}{4}$$

$$RS = \sum_{n=1}^{N} \sqrt{1-\Big(\underbrace{\tfrac{n}{\sqrt{2}N}}_{x_n}\Big)^2} \cdot \Delta x$$

$$\Delta x = \frac{b-a}{N} = \frac{1/\sqrt{2} - 0}{N} = \frac{1}{\sqrt{2}N}$$



Newton's Method for finding roots: $f(x) = 0$



$$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

<u>Prob</u>: Compute $\sqrt{2}$.

$f(x) = x^2 - 2$.

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^2 - 2)}{2x_n}$$

$$= \frac{x_n}{2} + \frac{1}{x_n}.$$

$x_0 = 2$.