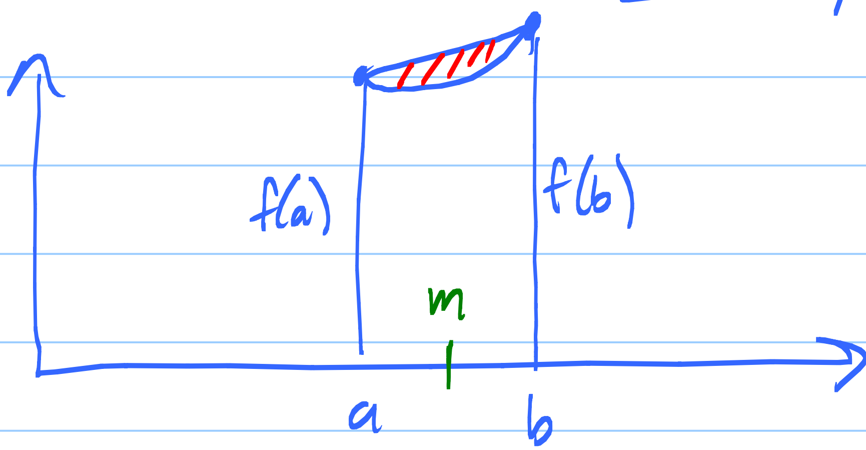
Error for the Trapezoid Rule $\frac{M(b-a)^3}{12N^2}$

where $M = \max_{[a,b]} |f''|$.

(Thanks to Purdue Prof. Chris Neugebauer and David Cruz-Uribe.)

First, find error for one trapezoid.



m = midpoint of a & b.

$m = \frac{a+b}{2}$.

$$E = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx$$

Calculation revolves around observation that

$$E = \int_a^b \underbrace{(x-m)}_{u} \underbrace{f'(x)\,dx}_{dv}$$

$u = x-m$

$du = dx$

$dv = f'(x)\,dx$

$v = f(x)$

Why:

$$= uv\Big|_a^b - \int_a^b v\,du$$

$$= (x-m)f(x)\Big|_a^b - \int_a^b f(x)\,dx$$

$$= \underbrace{\underbrace{(b-m)}_{\frac{b-a}{2}} f(b) - \underbrace{(a-m)}_{-\frac{b-a}{2}} f(a)}_{\frac{f(a)+f(b)}{2}(b-a)} - \int_a^b f(x)\,dx$$

$\leftarrow$ Trapezoid Area!

So $$E = \int_a^b (x-m)\underbrace{f'(x)}_{u}\,dx$$

Integrate by parts again.

$u = f'(x) \qquad dv = (x-m)\,dx$

$du = f''(x)\,dx \qquad v = \frac{1}{2}(x-m)^2$

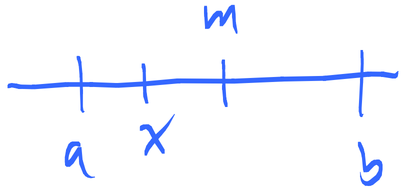
$$E = uv\Big|_a^b - \int_a^b v\,du = \frac{1}{2}(x-m)^2 f'(x)\Big|_a^b - \int_a^b f''(x)\frac{1}{2}(x-m)^2\,dx$$

$$= \underbrace{\frac{1}{2}(\underbrace{b-m}_{\frac{b-a}{2}})^2 f'(b) - \frac{1}{2}(\underbrace{a-m}_{-\frac{b-a}{2}})^2 f'(a)}_{\frac{1}{2}\left(\frac{b-a}{2}\right)^2 \underbrace{[f'(b)-f'(a)]}_{\frac{1}{2}\left(\frac{b-a}{2}\right)^2 \int_a^b f''(x)\,dx}} - \int_a^b \frac{1}{2}(x-m)^2 f''(x)\,dx$$

by Fund. Thm. Calc.

$$E = \int_a^b \left[\frac{1}{2}\left(\frac{b-a}{2}\right)^2 - \frac{1}{2}(x-m)^2\right] f''(x)\,dx$$

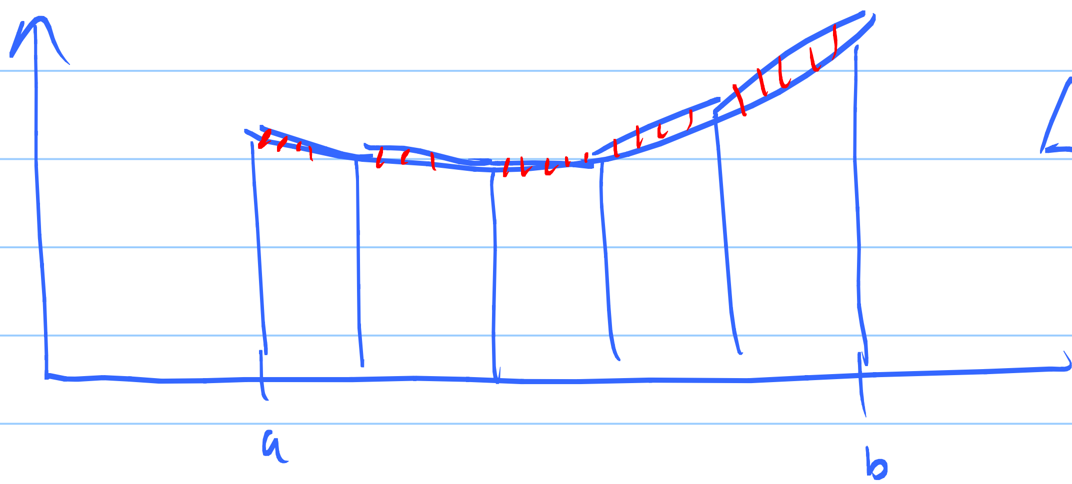
$\uparrow$ max of $\frac{1}{2}(x-m)^2$ on $[a,b]$.



$$|E| \le \int_a^b \underbrace{\left(\frac{1}{2}\left(\frac{b-a}{2}\right)^2 - \frac{1}{2}(x-m)^2 \right)}_{\ge 0,\ \text{so no absolute value here.}} |f''(x)|\, dx$$

$$\le M \underbrace{\int_a^b \frac{1}{2}\left(\frac{b-a}{2}\right)^2 - \frac{1}{2}(x-m)^2\, dx}_{\frac{1}{12}(b-a)^3} \quad \leftarrow \text{just do it.}$$

Error for one trapezoid is $\le \frac{M}{12}(b-a)^3$



$$\Delta x = \frac{b-a}{N}$$

Error from 1 $\le \frac{M}{12}\left(\frac{b-a}{N}\right)^3$

Sum of N errors $\le N \cdot \frac{M}{12}\left(\frac{b-a}{N}\right)^3 = \frac{M(b-a)^3}{12N^2}$.

HWK 8 on Home Page. Due Tuesday.

8.2 Bag of tricks for integrating trig functions.

<u>Trick #1</u>: Odd powers of sin and cos.

$$\int \sin^3 x \, dx$$

$$\int \underbrace{\sin^2 x}_{1-\cos^2 x} \; \underbrace{\sin x \, dx}_{-d(\cos x)}$$

$$\int (1-u^2)(-du) \qquad \underline{u=\cos x}$$

$$= -u + \frac{1}{3}u^3 + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

$$\int \cos^7 x \, dx$$

$$\int \underbrace{\cos^6 x}_{\substack{(\cos^2 x)^3 \\ (1-\sin^2 x)^3}} \; \underbrace{\cos x \, dx}_{d(\sin x)}$$

$$\int (1-u^2)^3 \, du \qquad \underline{u=\sin x}$$

$$= \int 1 - 3u^2 + 3u^4 - u^6 \, du$$

$$= \ldots$$

$$\int \sin^3 x \cos^{25} x \, dx$$

$$= \int \underbrace{\sin^2 x}_{1-\cos^2 x} \cos^{25} x \; \underbrace{\sin x \, dx}_{-d\cos x}$$

$$= \int (1-u^2)\, u^{25} (-du)$$

What to do with only even powers

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \frac{1}{2}(1 - \cos 2x) \, \frac{1}{2}(1 + \cos 2x) \, dx$$

$$= \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\theta = 2x$$

$$= \frac{1}{4} \int 1 - \frac{1}{2}(1 + \cos 4x) \, dx$$

$$= \frac{1}{4}\left[\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{4}\sin 4x\right] + C$$

Important Trig Identities:

$$\int \sin 3x \cos 5x \, dx = ?$$

$$\sin\alpha \sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

$$\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

$$\sin\alpha \cos\beta = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta)$$