$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$$\rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

plug $-\beta$ for β here to get

$$+ \quad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

8.2 Bag of tricks: Eliminating square roots.

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

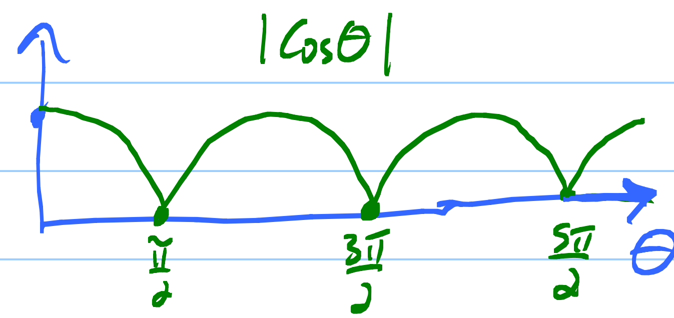
$x = 2\theta, \quad \theta = \frac{x}{2}$

$(1+\cos x) = 2\cos^2\frac{x}{2}$

EX: $I = \int_0^{\pi} \sqrt{1+\cos x}\, dx$

$$= \int_0^{\pi} \sqrt{2\cos^2\left(\frac{x}{2}\right)}\, dx$$

$$= \sqrt{2}\int_0^{\pi} \left|\cos\left(\frac{x}{2}\right)\right| dx$$



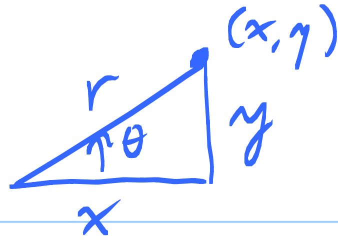
$\cos\frac{x}{2} > 0$ when $0 < x < \pi$.

$$= \sqrt{2}\int_0^{\pi} \cos\frac{x}{2}\, dx = \sqrt{2}\,\frac{1}{(1/2)}\sin\left(\tfrac{1}{2}x\right)\Big|_0^{\pi}$$

$$= 2\sqrt{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = 2\sqrt{2}.$$

Powers of Tan x and Sec x:

Trig identities:



$$x^2+y^2=r^2$$

Divide by r^2: $\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1$ $\quad\boxed{\cos^2\theta+\sin^2\theta=1}$

Divide by x^2: $1+\left(\frac{y}{x}\right)^2=\left(\frac{r}{x}\right)^2$ $\quad\boxed{1+\tan^2\theta=\sec^2\theta}$

EX: Even power of sec x:

$$\int \sec^4 x\,dx = \int \underbrace{\sec^4 x}_{\substack{(\sec^2 x)^2 \\ (1+\tan^2 x)^2}} \quad \underbrace{\sec^2 x\,dx}_{d(\tan x)}$$

$$= \int (1+u^2)^2\,du \quad \text{where } \underline{u=\tan x}$$

$$= \int 1+2u^2+u^4\,du = u+\tfrac{2}{3}u^3+\tfrac{1}{5}u^5+C$$

$$= \tan x+\tfrac{2}{3}\tan^3 x+\tfrac{1}{5}\tan^5 x+C$$

Same trick works for $\int \underbrace{\sec^6 x}_{\uparrow} \; \underbrace{\tan^{10} x}_{u^{10}}\,dx$

same as above.

EX: $\displaystyle\int \tan^6 x\,dx = \int (\tan^2 x)^3\,dx$

$$= \int (\sec^2 x - 1)^3 \, dx$$

$$= \int \sec^6 x - 3\sec^4 x + 3\sec^2 x - 1 \, dx$$

done above

easier than above

$3\tan x$

$-x$

Odd powers of Tan x and Sec x:

$$I = \int \sec^3 x \, dx = \int \underbrace{\sec x}_{u} \; \underbrace{\sec^2 x \, dx}_{dv}$$

$u = \sec x$ $\qquad$ $dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx$ $\qquad$ $v = \tan x$

$$I = uv - \int v \, du = (\sec x)(\tan x) - \int (\tan x) \sec x \tan x \, dx$$

$$= \sec x \tan x - \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec x \, dx$$

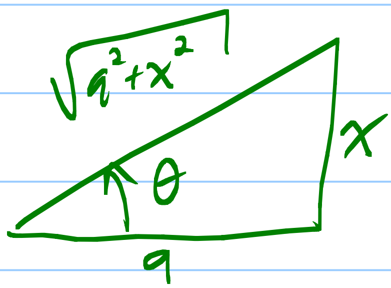
$$I = \sec x \tan x + \int \sec x \, dx - \underbrace{\int \sec^3 x \, dx}_{I}$$

Solve for I!

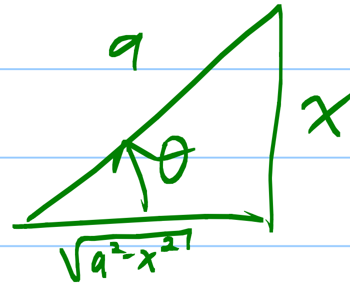
$$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx$$

$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$

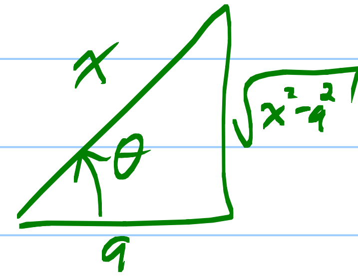
8.3 Trigonometric Substitution:



$x = a \tan\theta$



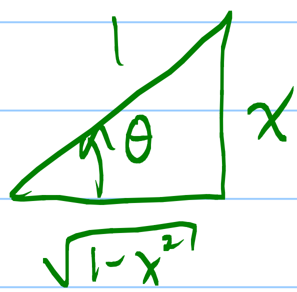
$x = a\sin\theta$



$x = a\sec\theta$

Key to substitution: choose $\triangle$ with proper $\sqrt{}$ term.

EX: $I = \int \sqrt{1-x^2}\, dx$ $\left[\sqrt{a^2-x^2},\ a=1\right]$



$\frac{x}{1} = \sin\theta$

$x = \sin\theta$

$\boxed{dx = \cos\theta\, d\theta}$

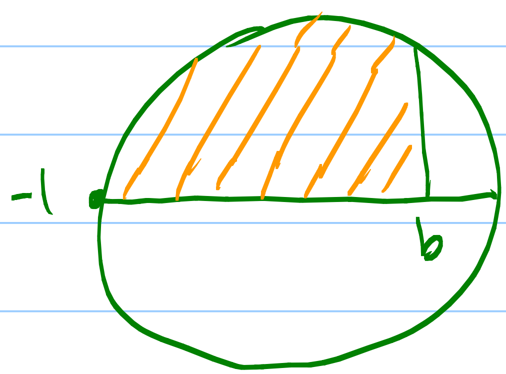
$\boxed{\sqrt{1-x^2} = \cos\theta}$

$$I = \int \underbrace{\sqrt{1-x^2}}_{\cos\theta}\ \underbrace{dx}_{\cos\theta\, d\theta} = \int \cos^2\theta\, d\theta$$

$$= \int \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta$$

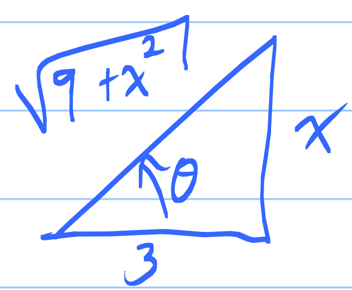
$$= \underbrace{\tfrac{1}{2}\theta}_{\theta = \sin^{-1} x} + \tfrac{1}{4} \underbrace{\sin 2\theta}_{2 \underbrace{\sin\theta}_{x} \underbrace{\cos\theta}_{\sqrt{1-x^2}}} + C$$

$$I = \tfrac{1}{2} \sin^{-1} x + \tfrac{1}{2} x \sqrt{1-x^2} + C$$



$$\text{Area} = \int_{-1}^{b} \sqrt{1-x^2}\, dx$$

EX: $I = \int \frac{dx}{\sqrt{9+x^2}}$ $\qquad a^2 = 9 \qquad a = 3$



$$\frac{\sqrt{9+x^2}}{3} = \sec\theta$$

$$x = a \tan\theta = 3 \tan\theta$$

$$\boxed{dx = 3 \sec^2\theta\, d\theta} \qquad \boxed{\sqrt{9+x^2} = 3 \sec\theta}$$

$$I = \int \frac{dx}{\sqrt{9+x^2}} = \int \frac{(3 \sec^2\theta\, d\theta)}{3 \sec\theta}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

Hmmm. But we saw that $\frac{d}{dx}\sinh^{-1}\frac{x}{3} = \frac{1}{\sqrt{9+x^2}}$ too! So $\sinh^{-1}\frac{x}{3} = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$ for some constant C. Plug in $x=0$ to determine C.

$$\underbrace{\sinh^{-1} 0}_{0} = \underbrace{\ln\left|1+0\right|}_{0} + C$$

So $C=0$.

<u>Remark</u>: Can do integrals in two different ways and get two different answers.
No problem! Answers must differ by a constant. Plus C takes care of it!