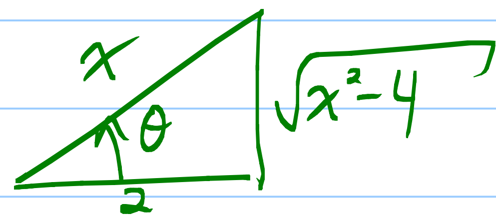
$$\int \sqrt{1-\sin x}\,dx = \int \sqrt{(1-\sin x)\left(\frac{1+\sin x}{1+\sin x}\right)}\,dx$$

$$= \int \sqrt{\frac{\overbrace{1-\sin^2 x}^{\cos^2 x}}{1+\sin x}}\,dx = \int \frac{\overbrace{\cos x\,dx}^{du}}{\sqrt{\underbrace{1+\sin x}_{u}}}$$

EX: $I = \int \frac{dx}{\sqrt{x^2-4}}$

$4 = a^2$

$a = 2$



$x = 2\sec\theta$

$$I = \int \frac{(2\sec\theta\tan\theta\,d\theta)}{2\tan\theta}$$

$$\begin{cases} dx = 2\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{cases}$$

$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

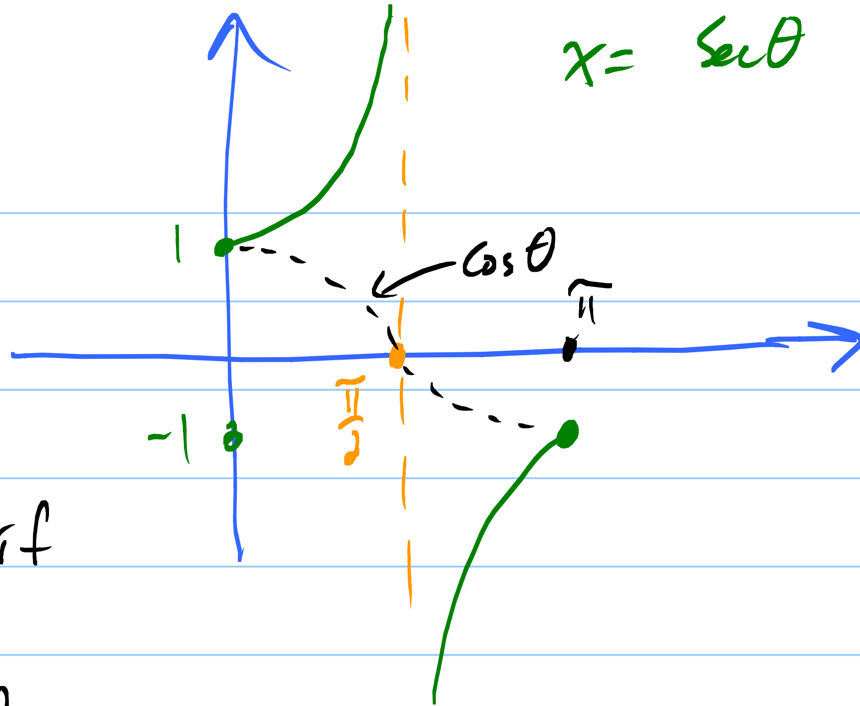
$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

Table

$$= \ln|x + \sqrt{x^2-4}| + C \quad ???$$

C can absorb the $-\ln 2$.

Subtle thing:



I did $x = 2\sec\theta$
where $0 < \theta < \frac{\pi}{2}$,
making $x > 2$,
Need to be careful if
I want $x < -2$.
Take $x = 2\sec\theta$,
$\frac{\pi}{2} < \theta < \pi$.

$$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2-4} = 2\sqrt{\sec^2\theta - 1}$$

$$= 2\sqrt{\tan^2\theta} = 2\,|\tan\theta|$$

$$= -2\tan\theta \text{ on } \frac{\pi}{2} < \theta < \pi.$$

Partial Fractions: How to do

$\int \frac{P(x)}{Q(x)}\,dx$ where P, Q polynomials.

Step 1: Cancel out common factors of P and Q.

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}$$

Step 2: Need $\deg P < \deg Q$. If this is not the case, do long division first.

EX: $$\int \frac{x^3+1}{x^2-9}\,dx = \int x + \frac{1+9x}{x^2-9}\,dx$$

$$\begin{array}{r|l} & x \\ \hline x^2-9 & x^3+1 \\ & x^3-9x \\ \hline & 1+9x \end{array} \leftarrow \text{remainder}$$

$$= \frac{1}{2}x^2 + \int \underbrace{\frac{9x+1}{x^2-9}}\,dx$$

now do partial fractions on this.

Step 3: Factor the denominator $Q(x)$ into terms $(x-r)^m$ $\leftarrow$ m = multiplicity of root

r = root

for real roots r, and $(x^2+Bx+C)^m$

terms that don't factor because of complex roots.

EX: $$Q(x) = \underbrace{x^3}_{(x-0)^3}(x-2)^2(x-3)\underbrace{(x^2+2x+2)}_{b^2-4ac<0}{}^3$$

$r=0$ is ok.

complex roots!

Step 4: Write out the Partial Fraction decomposition.

deg num < deg denom ✓

EX:
$$\frac{x^7+3x+10}{x^3(x-2)^2(x+3)(x^2+2x+2)^2(x^2+1)}$$

$$= \left[\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}\right]+\left[\frac{D}{(x-2)^2}+\frac{E}{(x-2)}\right]+\left[\frac{F}{x+3}\right]$$

$(x-0)^3$

$$+\left[\frac{Gx+H}{(x^2+2x+2)^2}+\frac{Ix+J}{(x^2+2x+2)}\right]$$

$$+\frac{Kx+L}{x^2+1}$$

Step 5: Multiply through by denominator:

$$\frac{x^2+x}{(x-2)(x^2+1)} = \frac{A}{x-2}+\frac{Bx+C}{x^2+1}$$

multiply by this

$$x^2+x = A(x^2+1)+(Bx+C)(x-2)$$

Step 6: Collect coefficients

$$\underbrace{x^2+x}_{1\cdot x^2+1\cdot x+0} = \underbrace{(A+B)}_{1}x^2 + \underbrace{(-2B+C)}_{1}x + \underbrace{(A-2C)}_{0}$$

Step 7: Equate coefficients:

x^2: $\quad 1 = A + B \qquad$ (a)

x: $\quad 1 = -2B + C \qquad$ (b)

const. term: $\quad 0 = A - 2C \qquad$ (c)

Step 8: Solve linear sytem for A, B, C, ...

Solve (b) for C: $\boxed{C = 1 + 2B}$ ← $C = 1 + 2(-1/5) = 3/5$

Plug into (c): $\quad 0 = A - 2(1+2B)$

$$\begin{cases} 2 = A - 4B \\ 1 = A + B \end{cases} \quad \text{(a)}$$

(−) $\quad 1 = -5B$

$\boxed{B = -1/5}$

$\boxed{C = 3/5}$

(a): $A = 1 - B = 1 - (-1/5) = \frac{6}{5}$

$\boxed{A = 6/5}$

Step 9: Do integral:

$$\int \frac{x^2+x}{(x-2)(x^2+1)}\,dx = \int \frac{6/5}{x-2} + \frac{(-1/5)x+(3/5)}{x^2+1}\,dx$$

$$= \frac{6}{5}\int \frac{dx}{\underbrace{x-2}_{u=x-2}} - \frac{1}{2\cdot 5}\int \frac{2x}{\underbrace{x^2+1}_{u}}\,dx + \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{6}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C$$