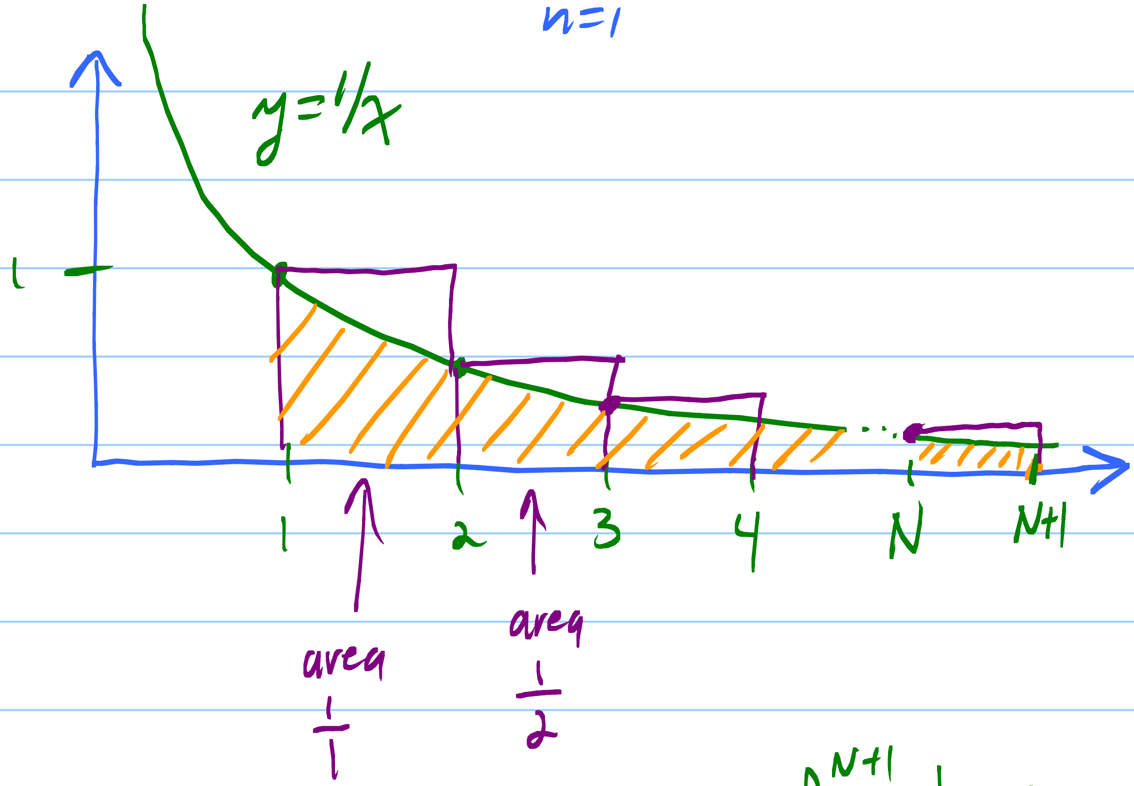
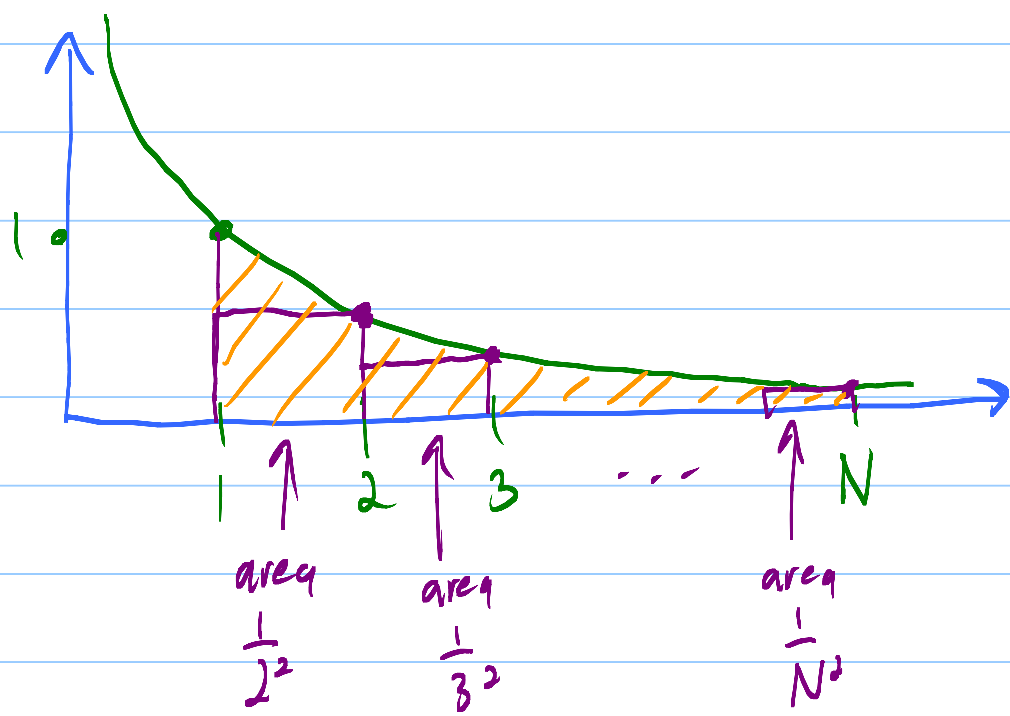
Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$



$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > \int_1^{N+1} \frac{1}{x}\,dx = \ln(N+1)$$

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge to a finite number?



See $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2} < \int_1^N \frac{1}{x^2}\, dx$

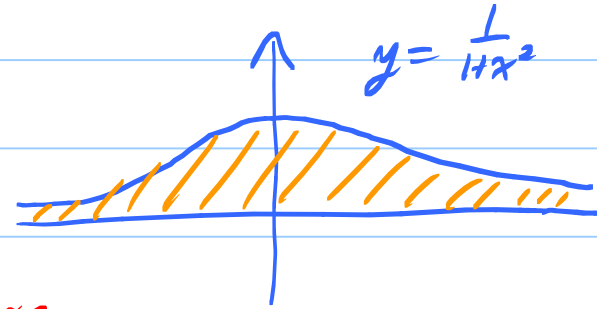
Add $\frac{1}{1^2}$ to each side:

$$\sum_{n=1}^{N} \frac{1}{n^2} < 1 + \underbrace{\left[\frac{1}{-2+1} x^{-2+1}\right]_1^N}_{2-\frac{1}{N}} < 2$$

Real number fact: An increasing sequence that is bounded above has a limit.

Famous fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (≤ 2).

EX: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi$



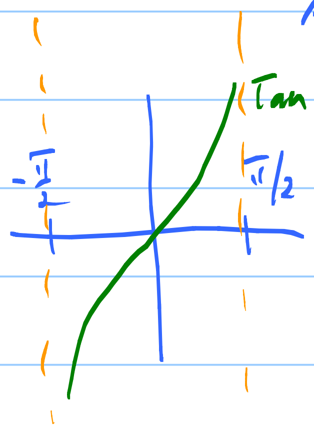
Warning: Need $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ to make sense in order to talk about $\int_{-\infty}^{\infty}$.

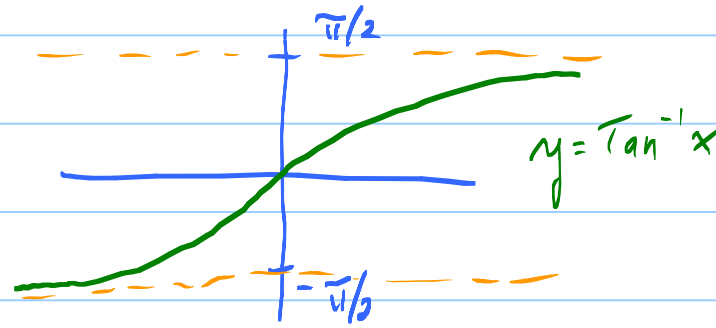
EX: $\int_{-\infty}^{\infty} x\, dx = ?$ $\qquad \int_{-A}^{A} x\, dx = 0$

$$\int_{-2A}^{3A} x\, dx = \frac{1}{2}(3A)^2 - \frac{1}{2}(-2A)^2 = \text{BIG as } A \to \infty.$$

$$\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx = \lim_{A\to-\infty}\int_{A}^{0}\frac{1}{1+x^2}\,dx$$

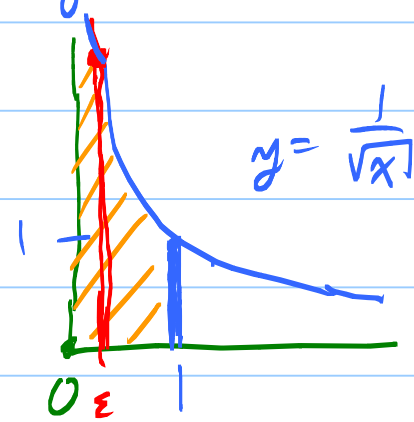
$$= \lim_{A\to-\infty}\left[\tan^{-1}x\right]_{A}^{0} = \underbrace{\tan^{-1}0}_{0} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$





Similarly $\displaystyle\int_{0}^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi \quad \checkmark$$



EX: $\displaystyle\int_{0}^{1}\frac{1}{\sqrt{x}}\,dx$

$$= \lim_{\varepsilon\to0+}\int_{\varepsilon}^{1} x^{-1/2}\,dx$$

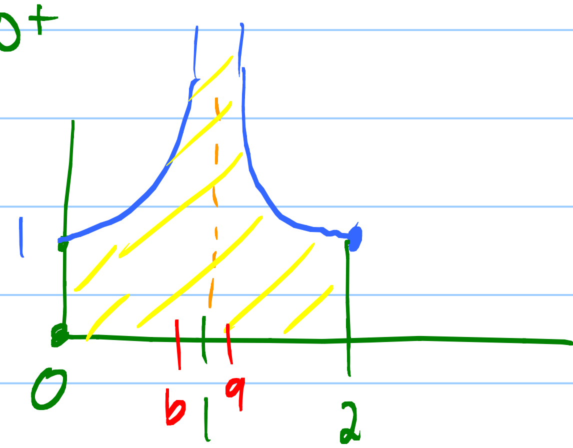
$$= \lim_{\varepsilon\to0+}\left[\frac{1}{-\frac{1}{2}+1}\,x^{-1/2+1}\right]_{\varepsilon}^{1} = \lim_{\varepsilon\to0+}\left[2\sqrt{1}-2\sqrt{\varepsilon}\right]$$

$$= 2 \longleftarrow \int \text{ converges. (finite)}$$

EX: $\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+}\left[\ln 1 - \ln\varepsilon\right]$

$= \lim_{\varepsilon\to 0^+} \ln\frac{1}{\varepsilon} = \infty.$ $\int$ diverges

EX: $I = \int_0^2 \frac{1}{(x-1)^{2/3}}\,dx$



Can't use Fund. Thm. Calc.!

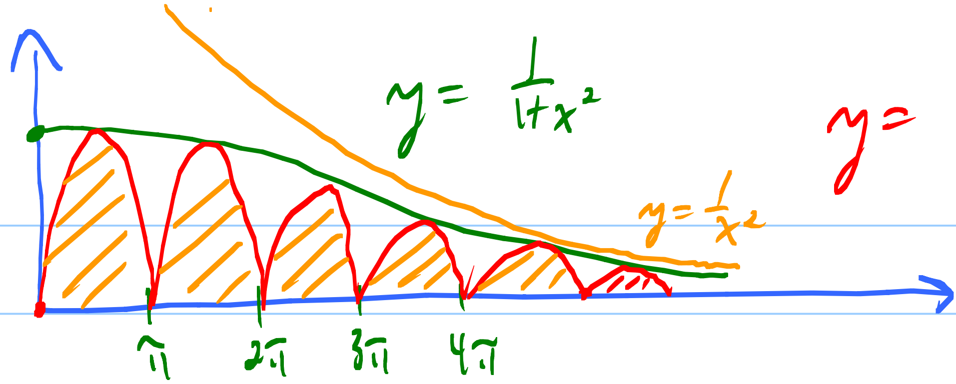
$$I = \lim_{b\to 1^-}\int_0^b (x-1)^{-2/3}\,dx + \lim_{a\to 1^+}\int_a^2 (x-1)^{-2/3}\,dx$$

$$= \lim_{b\to 1^-}\left[3(x-1)^{1/3}\right]_0^b + \lim_{a\to 1^+}\left[3(x-1)^{1/3}\right]_a^2$$

$$= \lim_{b\to 1^-}\left[3(b-1)^{1/3} - 3(-1)\right] + \lim_{a\to 1^+}\left[3 - 3(a-1)^{1/3}\right]$$

$$= [0+3] + [3-0] = 6$$

Prob: Is $\int_0^\infty \frac{\sin^2 x}{1+x^2}\,dx$ finite?



$$\frac{\sin^2 x}{1+x^2} \le \frac{1}{1+x^2} < \frac{1}{x^2}$$

Tail end of area $\int_a^\infty \frac{\sin^2 x}{1+x^2}\,dx < \underbrace{\int_a^\infty \frac{1}{x^2}\,dx}_{=1/a}$

finite $\qquad\qquad$ finite.

(Can take $a = 1000$.)

<u>Comparison Test:</u> $0 \le f(x) \le g(x)$ on $[a, \infty)$.

$$\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$$

$\int g\,dx$ finite $\implies$ $\int f\,dx$ finite too.

$\int f\,dx$ infinite $\implies$ $\int g\,dx$ infinite too.

<u>Limit Comparison Test:</u> $f(x) \ge 0$, $g(x) \ge 0$.

Suppose $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$.

Then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ do same thing. [They either both converge or they both diverge.]

EX: Is $\displaystyle\int_0^\infty \frac{x}{x^3+7}\,dx$ finite?

Think $\displaystyle\frac{x}{x^3+7} \sim \frac{x}{x^3} = \frac{1}{x^2}$ for big x.

$$\lim_{x\to\infty} \frac{\left(\frac{x}{x^3+7}\right) \leftarrow f(x)}{\left(\frac{1}{x^2}\right) \leftarrow g(x)} = \lim_{x\to\infty} \frac{x^3}{x^3+7} = \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$\lim_{x\to\infty} \left(\frac{1}{1+\frac{7}{x^3}}\right) = \frac{1}{1+0} = 1$$

$0 < L < \infty$.

Tail end of $\frac{1}{x^2}$: $\displaystyle\int_a^\infty \frac{1}{x^2}\,dx < \infty$ if $a > 0$.

So $\displaystyle\int_0^\infty \frac{x}{x^3+7}\,dx$ is finite too.