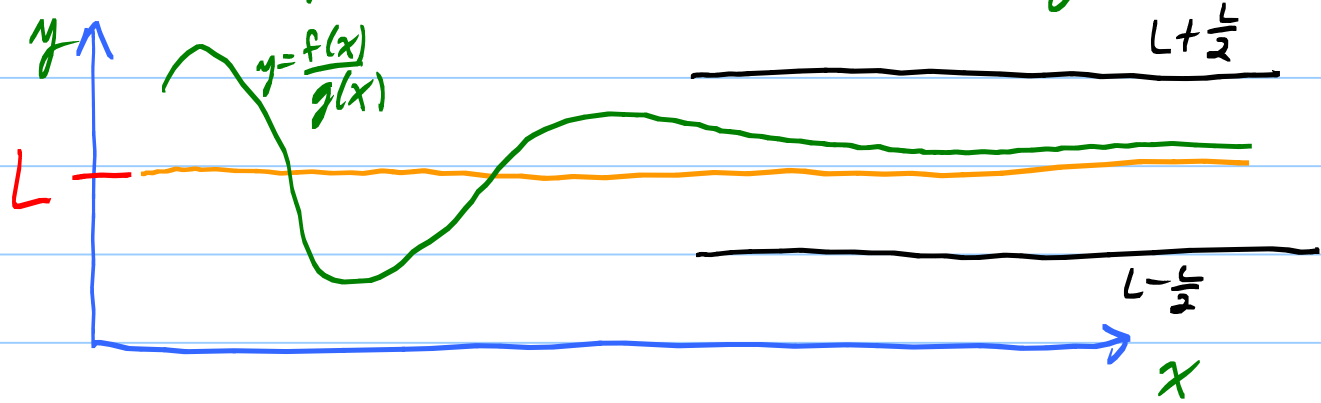
Limit Comparison Test: $f(x) \geq 0$, $g(x) \geq 0$ for $x \geq a$.

If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$, Then

$\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ do the same thing.

Why:



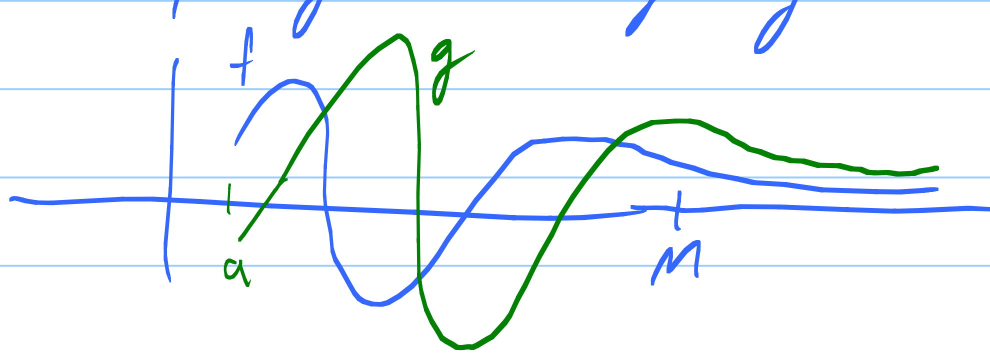
Think $\varepsilon = \frac{L}{2}$ in the definition of limit. See there is an M such

$$\frac{L}{2} < \frac{f(x)}{g(x)} < \frac{3L}{2} \quad \text{when } x \geq M.$$

Now $\frac{L}{2} g(x) < f(x) < \frac{3L}{2} g(x)$ when $x \geq M$.

So $\frac{L}{2} \int_M^\infty g(x)\,dx < \int_M^\infty f(x)\,dx < \frac{3L}{2} \int_M^\infty g(x)\,dx$.

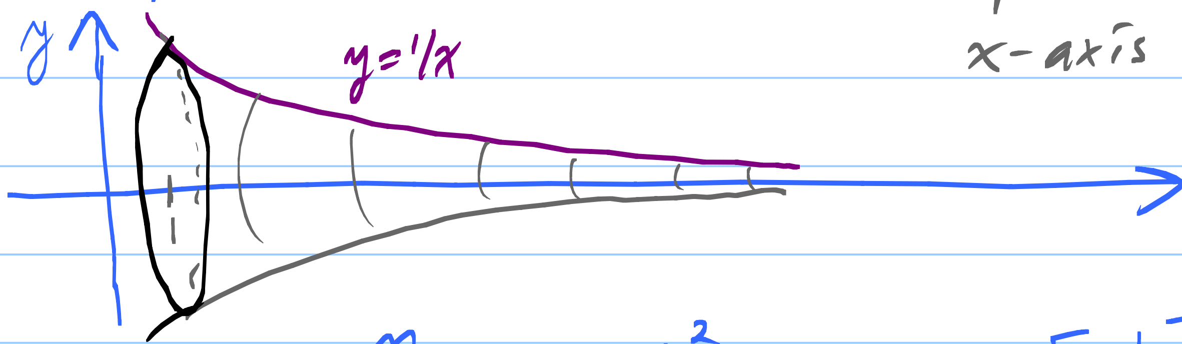
Note: Can always ignore the beginning of these integrals.



EX: Compare to $\frac{1}{x^p}$, but need $M > 0$ when you do that. The tail end determines convergence of $\int_a^\infty \cdots \, dx$.

p. 506: 76. Gabriel's horn

Spin around the x-axis from 1 to ∞.



$$V = \text{Volume} = \int_1^\infty \pi \left[\frac{1}{x}\right]^2 dx = \pi \left[-\frac{1}{x}\right]_1^\infty = \pi(0-(-1)) = \pi$$

$$S = \text{Surface Area} = \int_1^\infty 2\pi y \, ds = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\, dx$$

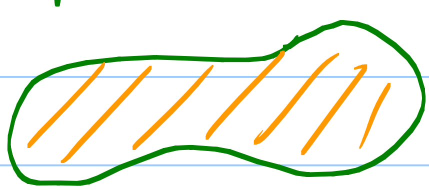
Comparison Test: $\frac{1}{x}\sqrt{1+\frac{1}{x^4}} > \frac{1}{x}$ when $x \geq 1$

$$\int_1^\infty \frac{1}{x} dx = \left[\ln x\right]_1^\infty = \infty.$$

So S is infinite! (Think can of paint here!)

<u>Isoperimetric Inequality</u>: Circle $2\pi r = L$

$$\text{Area} = \pi r^2 = \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$



L

$$\text{Area} \leq \frac{L^2}{4\pi}.$$

Laplace Transform: $\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\,dt = F(s)$

Fact: $\mathcal{L}[f'(t)] = sF(s) - f(0)$

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$

EX: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ $\qquad \mathcal{L}[\cos bt] = \frac{s}{s^2+b^2}$.

Exciting idea: $\qquad ay'' + by' + cy = f(t)$

Hit with $\mathcal{L}$: $\quad a[s^2Y - sy(0) - y'(0)] + b[sY - y(0)] + cY = F(s)$

(Here $Y(s) = \mathcal{L}[y(t)]$.) $\qquad$ Now solve for Y.

Get solution $y = \mathcal{L}^{-1}[Y]$.

EX: $\quad y'' + 3y' + 2y = e^{-3t}$ $\qquad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$s^2Y + 3sY + 2Y = \frac{1}{s+3}$$

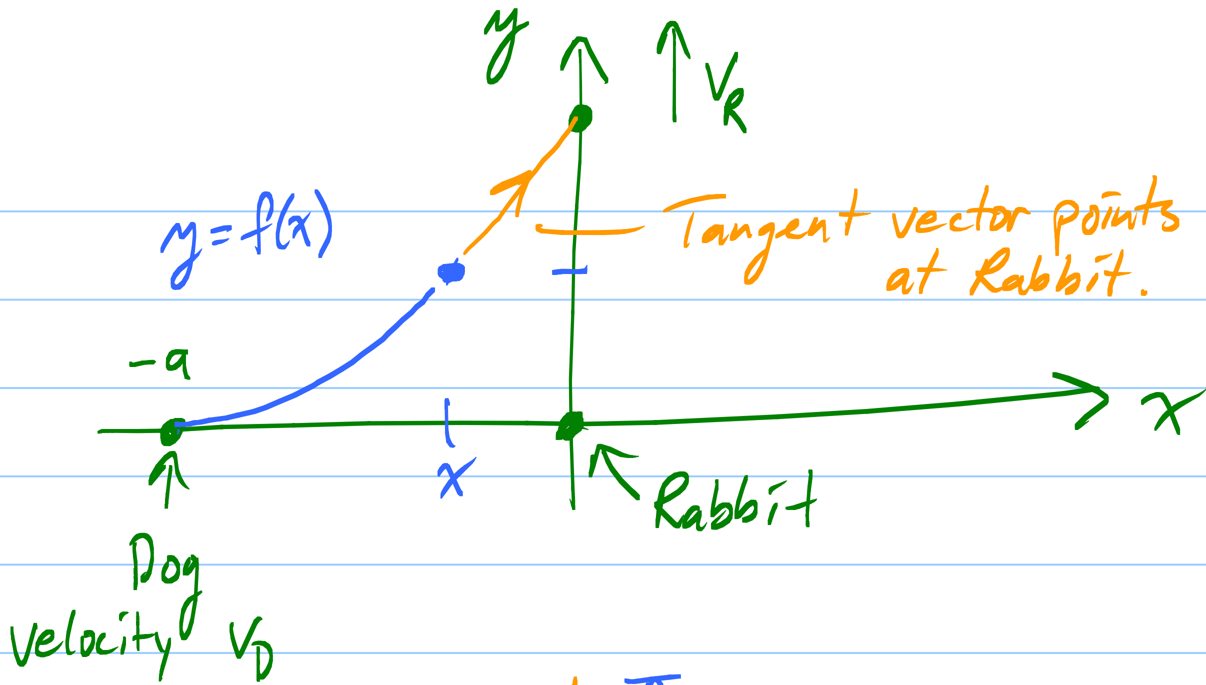
$$Y = \frac{1}{s^2+3s+2} \cdot \frac{1}{s+3} = \frac{1}{(s+1)(s+2)(s+3)}$$

$$Y = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

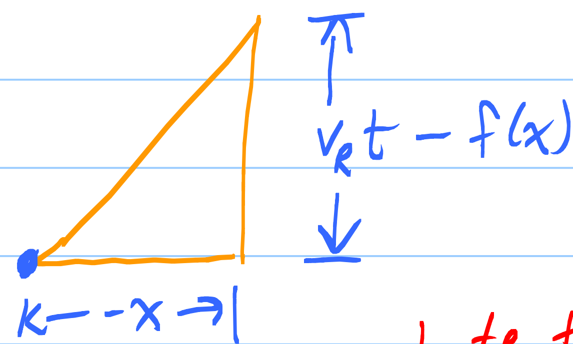
Partial fractions!

Get $\quad y = Ae^{-t} + Be^{-2t} + Ce^{-3t}$.

Prob:



Slope $= f'(x)$



hate the t!

So $\quad f'(x) = \dfrac{v_R t - f(x)}{-x}$

Trick: $\quad v_D t = \text{Length of path} = \displaystyle\int_{-a}^{x} \sqrt{1 + f'(u)^2}\, du$

Now $\quad f'(x) = \dfrac{\frac{v_R}{v_D} \int_{-a}^{x} \sqrt{1+f'(u)^2}\, du - f(x)}{-x}$

Solve for the integral:

$$\frac{v_R}{v_D} \int_{-a}^{x} \sqrt{1 + f'(u)^2}\, du = f(x) - x f'(x).$$

hate this $\int$. Diff'tiate.

Fund. Thm. Calc. :

$$\frac{V_R}{V_D}\sqrt{1+f'(x)^2} = -x\,f''(x).$$

An ODE! Let $v = f'(x)$.

$$\frac{V_R}{V_D}\sqrt{1+v^2} = -x\frac{dv}{dx}$$ Separable!

$$-\frac{V_R}{V_D}\frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$$

$$-\frac{V_R}{V_D}\int\frac{dx}{x} = \int\frac{dv}{\sqrt{1+v^2}}$$

$$-\frac{V_R}{V_D}\ln|x| = \sinh^{-1} v + C.$$

Take $a=1$. Plug in $x=-a=-1$.

$v = \frac{dy}{dx} = 0$ at time 0 when $x=-1$.

Get $C=0$.

$$v = \sinh\left(-\frac{V_R}{V_D}\ln|x|\right)$$

$$y = \int \sinh\left(\ln|x|^{-\underbrace{V_R/V_D}_{p}}\right) dx + C$$

$$= \int \frac{e^{\ln|x|^{-p}} - e^{-\ln|x|^{-p}}}{2} \, dx$$

$$= \int \frac{|x|^{-p} - |x|^{p}}{2} \, dx$$

$$y = \int \frac{(-x)^{-p} - (-x)^{p}}{2} \, dx$$

<u>Case</u> $V_R = V_D$. $\quad y = \frac{1}{2}\int -\frac{1}{x} - (-x) \, dx$

$$= -\frac{1}{2}\ln|x| + \frac{x^2}{2} + C$$

When $x=-1$, $y=0$. Get $C = -\frac{1}{2}$.