p. 505: 65 a $\quad I = \int^2 \frac{dx}{x(\ln x)^p} \qquad u = \ln x$

As x goes from 1 to 2,
u goes from 0 to $\ln 2$.

$$I = \int_0^{\ln 2} \frac{1}{u^p}\, du = \lim_{A \to 0+} \int_A^{\ln 2} \frac{1}{x^p}\, dx$$

x^{-p}

$p = 1$. Bad.

$p \neq 1$: $\quad I = \lim_{A \to 0+} \left[\frac{1}{-p+1} x^{-p+1} \right]_A^{\ln 2}$

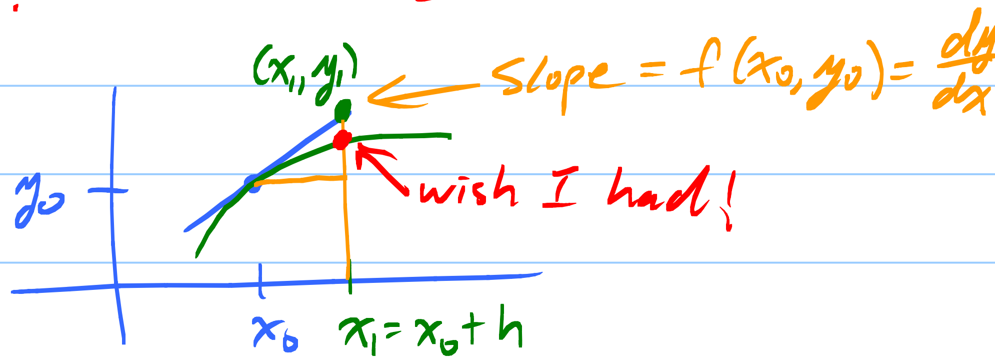
Need $-p+1 > 0$ to get convergence (finite I).

9.1 First Order Ordinary Diff. Eqns:

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0.$$

EX: $\quad \underbrace{\frac{dy}{dx} = x^2 + y^2}_{\text{O.D.E.}}, \quad \underbrace{y(0) = 0.}_{\text{I.V.P}}$

Euler Method:



$$\text{Slope} = f(x_0, y_0) = \frac{y_1 - y_0}{h}$$

Get $y_1 = y_0 + f(x_0, y_0)\, h.$

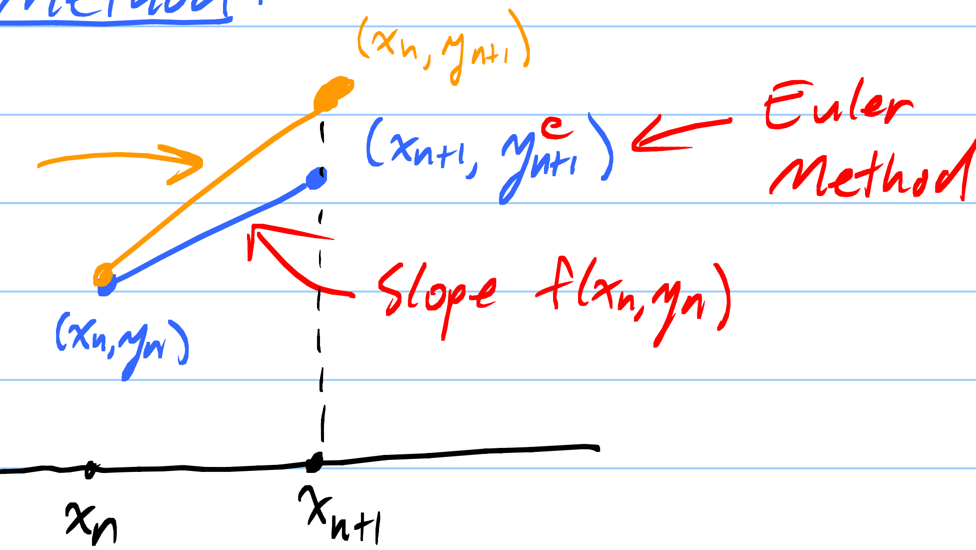
Repeat: $\boxed{y_{n+1} = y_n + f(x_n, y_n)\, h}$

$$x_{n+1} = x_n + h.$$

Improved Euler Method:

$$\frac{f(x_n, y_n) + f(x_{n+1}, y^e_{n+1})}{2}$$

Improved slope guess.



$$x_{n+1} = x_n + h$$

$$y^e_{n+1} = y_n + f(x_n, y_n)\, h$$

$$y_{n+1} = y_n + \left[\frac{f(x_n, y_n) + f(x_{n+1}, y^e_{n+1})}{2}\right] h$$

Fact: Use these methods to solve $\frac{dy}{dx} = f(x),$ $y(a) = y_0.$

Sol^n is $y(x) = y_0 + \int_a^x f(t)\, dt$

Euler's Method: Rectangle Rule with left endpoints.
Improved Euler's: Trapezoid Rule!
Simpson's: ? Runge-Kutta Method.

Getting formulas for solutions.

Easiest: $\frac{dy}{dx} = f(x)$: $\quad y = \int f(x)\,dx$

Next: $\frac{dy}{dx} = ky$: $\quad y = Ce^{kx}$

Separable: $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ $\quad \int g(y)\,dy = \int f(x)dx + C$

9.2 First Order Linear ODEs.

EX: $x^2 \frac{dy}{dx} + 2x\,y = x^3, \quad y(1) = 2.$

Linear in $\frac{dy}{dx}$ and y.

$$\frac{d}{dx}\left(x^2 y\right) = x^3$$

$$x^2 y = \int x^3\,dx = \frac{1}{4}x^4 + C$$

$y = \frac{1}{4}x^2 + \frac{c}{x^2}$ Need $y(1) = 2$

$y(1) = \frac{1}{4} \cdot 1^2 + \frac{c}{1^2} \overset{\text{want}}{=} 2$. $c = 2 - \frac{1}{4} = \frac{7}{4}$

$y = \frac{1}{4}x^2 + \frac{(7/4)}{x^2}$.

General Method: $\alpha(x)\frac{dy}{dx} + \beta(x)y = \gamma(x)$

Step 1: Divide the eqn by $\alpha(x)$ to put in

Standard Form: $\frac{dy}{dx} + P(x)y = Q(x)$

Step 2: Multiply by an "integrating factor" $u(x)$:

$$\underbrace{u(x)\frac{dy}{dx} + u(x)P(x)y}_{} = u(x)Q(x).$$

Wish $\frac{d}{dx}(u\,y) = u\frac{dy}{dx} + u'y$

Step 3: See that we need $\frac{du}{dx} = uP$. Separable!

$$\frac{du}{u} = P\,dx$$

$$\int \frac{du}{u} = \int P\,dx$$

$$\ln u = \int P\,dx$$

Can assume u is positive. Otherwise use $-u$ in place of u.

(Note: Skip $+C$ in the $\int$. I just want one u that works.)

$$\boxed{u = e^{\int P(x)\,dx}}$$

Step 4: Do it.

$$\frac{dy}{dx} + Py = Q$$

$$\underbrace{u\left(\frac{dy}{dx} + Py\right)} = uQ$$

$$\frac{d}{dx}(uy) = uQ$$

$$uy = \int uQ\,dx + C$$

$$y = \frac{1}{u}\int uQ\,dx + \frac{C}{u}$$

EX: $2\frac{dy}{dx} - 4y = 6e^x$

Step 1: Stand. Form $\quad \frac{dy}{dx} - 2y = 3e^x$

$P(x) = -2$

Step 2: $u = e^{\int P(x)\,dx} = e^{\int -2\,dx} = e^{-2x}$

Step 3:

$$\underbrace{e^{-2x}\left(\frac{dy}{dx} - 2y\right)} = e^{-2x}(3e^{x})$$

$$\frac{d}{dx}\left(e^{-2x}y\right) = 3e^{-x}$$

$$e^{-2x}y = \int 3e^{-x}\,dx = -3e^{-x} + C$$

$$\boxed{y = -3e^{x} + Ce^{2x}}$$