Linear ODEs:

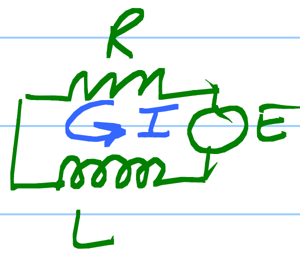
$$L[y] = ay'' + by' + cy$$

Homogeneous Eqn: $L[y] = 0$.

Called "linear" because $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$

Consequence: If y_1 and y_2 solve to $L[y]=0$, so does $c_1 y_1 + c_2 y_2$.

Electric circuits:



$$RI + L\frac{dI}{dt} = E$$ ← Linear First order ODE!

Ex: $2\frac{dI}{dt} + 4I = 8\sin\omega t$

Standard Form: $\frac{dI}{dt} + \underset{\uparrow\, P(t)=2}{2}I = 4\sin\omega t$

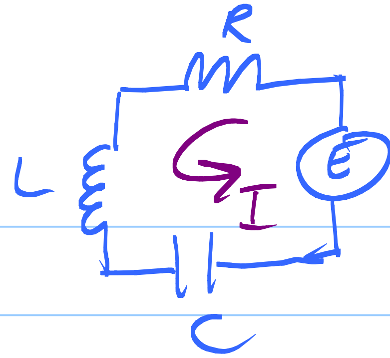
$$u(t) = e^{\int P\,dt} = e^{\int 2\,dt} = e^{2t}$$

$$\underbrace{e^{2t}\left(\frac{dI}{dt} + 2I\right)}_{\frac{d}{dt}\left[e^{2t} I\right]} = 4e^{2t}\sin\omega t$$

$$e^{2t} I = \int 4e^{2t}\sin\omega t\,dt + C$$

$I(0) = I_0$ determines the C.

Fancier circuit:

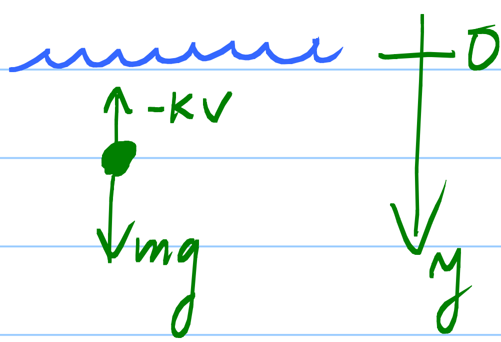


$$RI + L\frac{dI}{dt} + \frac{1}{C}Q = E. \quad \text{But } \frac{dQ}{dt} = I.$$

Differentiate: $R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}I = E'(t)$

2nd order linear ODE.

__EX__:

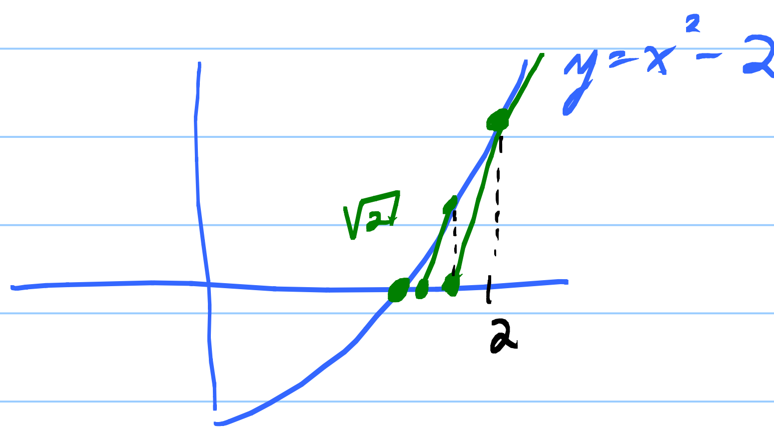


$F = ma$

$$\boxed{mg - kv = m\frac{dv}{dt}}$$

First order linear.

10.1 Sequences.



$x_1 = 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

See from picture

$\sqrt{2} < \cdots < x_4 < x_3 < x_2 < x_1$

Fun exercise: Show this using inequalities.

Fact: A decreasing sequence that is bounded from below must have a limit. (Completeness.)

So $x_n \to L$, a limit where $L \geq \sqrt{2}$?.

Claim: $L^2 = 2$.

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

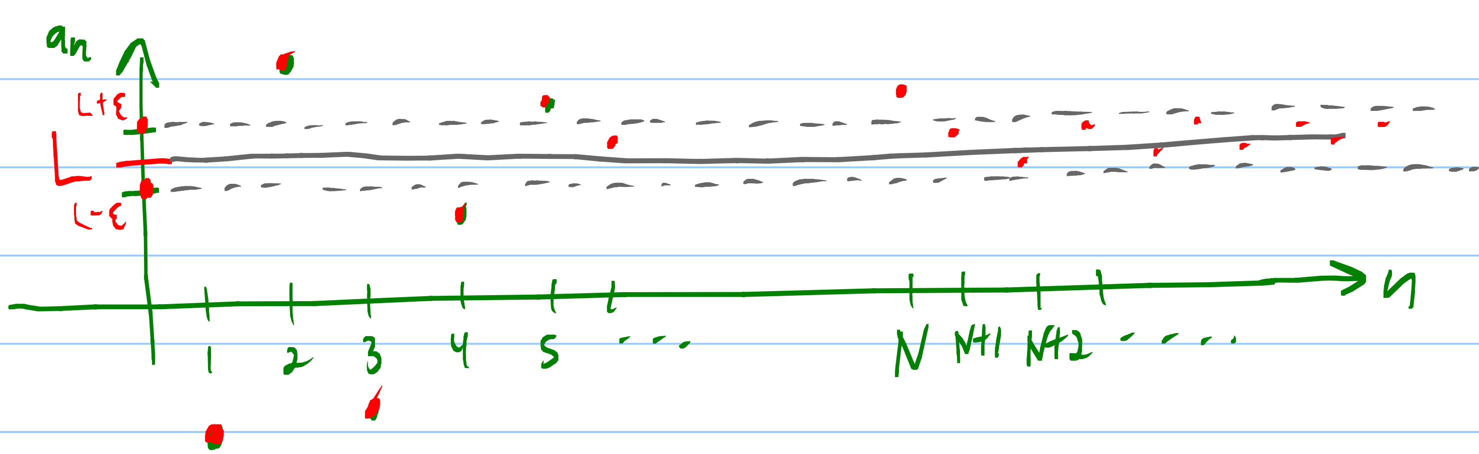
$\downarrow$ as $n \to \infty$.

$$L = \frac{L}{2} + \frac{1}{L}$$

$$\frac{L}{2} = \frac{1}{L}$$

$$L^2 = 2 \checkmark$$

Def^n: $\lim_{n\to\infty} a_n = L$ means: Given any $\varepsilon > 0$, no matter how small, there is an N such that $|a_n - L| < \varepsilon$ when $n > N$.



Def^n: $\lim_{n\to\infty} a_n = \infty$ means:

Given $M>0$, no matter how big, there is an N such that $a_n > M$ when $n > N$.

Famous sequences: $\lim\limits_{n\to\infty}\left(1+\frac{b}{n}\right)^n = e^b$

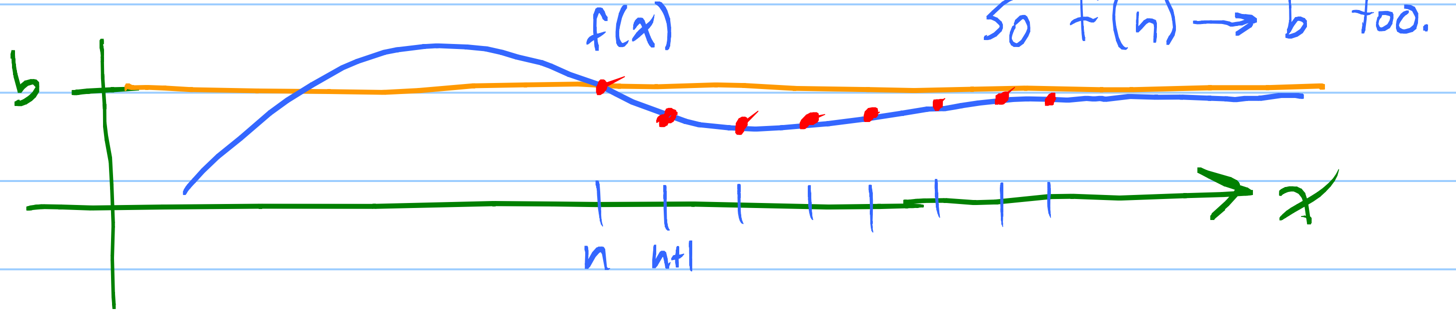
$$a_n = \left(1+\frac{b}{n}\right)^n$$

$$\ln a_n = \ln\left(1+\frac{b}{n}\right)^n = n\ln\left(1+\frac{b}{n}\right) = \frac{\ln\left(1+\frac{b}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$f(x) = \frac{\ln\left(1+\frac{b}{x}\right)}{\left(\frac{1}{x}\right)}$$

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{b}{x}\right)}{\left(\frac{1}{x}\right)} \;\overset{\left(\frac{0}{0}\right)}{\underset{L'H}{=}}\; \lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{b}{x}\right)}\cdot\left(-\frac{b}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{b}{1+\frac{b}{x^2}} = \frac{b}{1+0} = b$$



See that $\ln a_n \to b$ as $n\to\infty$.

$$a_n = e^{\ln a_n} \longrightarrow e^b \quad \text{as } n\to\infty. \checkmark$$

Fact: If $f(x)$ is a function such that $\lim_{x\to A} f(x) = L$ and x_n is sequence such that $\lim_{n\to\infty} x_n = A$, then $\lim_{n\to\infty} f(x_n) = L$.

EX: $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Similar.

EX: $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

$$\frac{1\cdot 2\cdot 3\cdot 4 \cdots (n-1)\cdot n}{n\cdot n\cdot n\cdot n \cdots n\cdot n} < \frac{1}{n}\cdot \underbrace{1\cdot 1\cdot 1 \cdots 1}_{n-1 \text{ 1's}}$$

$< \varepsilon$ when $n > \frac{1}{\varepsilon}$. $\varepsilon = 10^{-6}$. Need $n > 10^6$.

EX: $\lim_{n\to\infty} \frac{100^n}{n!}$

$$\frac{100\cdot 100\cdot 100\cdot 100\cdot 100 \cdots 100\cdot 100}{1\cdot 2\cdot 3\cdot 4\cdot 5 \cdots (n-1)\cdot n}$$

$$\frac{100\cdot 100\cdot 100 \cdots 100\cdot 100 \cdots 100\cdot 100}{\underbrace{1\cdot 2\cdot 3 \cdots 99}_{B}\cdot 100 \cdots (n-1)\cdot n}$$

$$< B \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \left(\frac{100}{n}\right) < \underbrace{10^{-6}}_{\varepsilon}$$

if $\quad n > 10^{6} \cdot 100 \cdot B$

$$n > \frac{1}{\varepsilon} \cdot 100 \cdot B$$