Exam 2 in class Wed., Nov. 16

Sandwich Theorem: Sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $a_n \le b_n \le c_n$ for $n > N_0$. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

EX: $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\downarrow 0 \qquad \downarrow 0$

$\{a_n\}_{n=1}^{\infty}$ Sequence notation.

Conclude $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Fact: If $a > 0$, then $\lim_{n\to\infty} a^{1/n} = 1$.

Why: $a^{1/n} = e^{\ln a^{1/n}} = e^{\frac{1}{n}\ln a} \xrightarrow[n\to\infty]{} e^0 = 1$

by continuity of e^x.

Fact: x is real. $\lim_{n\to\infty} x^n = ?$

Cases: $x > 0$. $x^n = e^{n\ln x}$

Case a: $0 < x < 1$

Then $\ln x < 0$.

$e^{n\ln x} \to 0$ as $n\to\infty$.

Case b: $x=1$. $x^n=1$. Converges.

Case c: $x>1$. $\ln x>0$. $e^{n\ln x}\to\infty$ as $n\to\infty$.

Case $x=0$: $x^n=0$. Converges.

Case $x<0$: $x^n=(-x)^n=(-1)^n(-x)^n$

$$=(-1)^n e^{n\ln(-x)}$$

Case a: $0<-x<1$. $x^n\to 0$

Case b: $x=-1$. $x^n=(-1)^n$. diverges.

Case c: $-x>1$. $(-1)^n(-x)^n$ diverges bad.

Summary: $\lim\limits_{n\to\infty} x^n = 0$ if $|x|<1$.

diverges if $|x|>1$.

1 if $x=1$

diverges if $x=-1$

EX: $\sqrt[n]{2n}=\sqrt[n]{2}\sqrt[n]{n}\to 1\cdot 1$ as $n\to\infty$.

EX: $$\left(\sqrt{n^2+100}-n\right)\cdot\frac{\sqrt{n^2+100}+n}{\sqrt{n^2+100}+n}$$

$$=\frac{(n^2+100)-n^2}{\sqrt{n^2+100}+n}=\frac{100}{\sqrt{n^2+100}+n}<\frac{100}{n}\to 0$$

as $n\to\infty$.

So $\lim\limits_{n\to\infty}\left(\sqrt{n^2+100}-n\right)=0$.

EX: $\operatorname{Tanh}(\ln n) = \dfrac{e^{\ln n} - e^{-\ln n}}{e^{\ln n} + e^{-\ln n}} = \dfrac{n - \frac{1}{n}}{n + \frac{1}{n}} \cdot \dfrac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$

$$= \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n^2}} \to \frac{1-0}{1+0} = 1 \quad \text{as } n \to \infty.$$

EX: $\sqrt[n]{n} = a_n \qquad \ln a_n = \ln n^{1/n} = \frac{1}{n} \ln n$

$$= \frac{\ln n}{n}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \underset{L'H}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$$

So $\lim_{n\to\infty} \ln a_n = 0$. But $a_n = e^{\ln a_n}$.

So $a_n = e^{\ln a_n} \to e^0 = 1$ as $n \to \infty$

by continuity of e^x.

10.2 Series.

Important $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

<u>Geometric Series</u>:

$$\underbrace{a + ar + ar^2 + ar^3 + \cdots + ar^N}_{\sum_{n=0}^{N} ar^n} = S_N$$

S_N: partial sum

$$\sum_{n=0}^{\infty} ar^n = L \quad \text{means} \quad S_N \to L \text{ as } N \to \infty.$$

Trick:

$$S_N = a + ar + ar^2 + \cdots + ar^N$$

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$$rS_N = \quad ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1}$$

$$(1-r)S_N = a - ar^{N+1}$$

$$S_N = \frac{a}{1-r} - \underbrace{\frac{ar^{N+1}}{1-r}}_{\to 0 \text{ as } N \to \infty \text{ if } |r|<1.}$$

Big Fact: Geometric series converge if $|r|<1$. Diverge otherwise.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r|<1.$$

Consequence: Repeating decimals are rational numbers.

EX: $0.123123123\overline{123}$

$= 123 \cdot 10^{-3} + 123 \cdot 10^{-6} + 123\, 10^{-9} + \cdots$

$= 123 \cdot 10^{-3} \left[1 + (10^{-3}) + (10^{-3})^2 + \cdots\right]$

$= 123 \cdot 10^{-3} \cdot \dfrac{1}{1-10^{-3}} = \dfrac{123}{999}$

EX: $\dfrac{e}{\pi} + \dfrac{e}{\pi^2} + \dfrac{e}{\pi^3} + \cdots = S$

$e + e\left(\frac{1}{\pi}\right) + e\left(\frac{1}{\pi}\right)^2 + \cdots = S + e$

$a = e. \quad r = 1/\pi < 1.$

$$\frac{e}{1-\frac{1}{\pi}} = S + e$$

$$S = \frac{\pi e}{\pi - 1} - e = \frac{e}{\pi - 1}$$

Bouncing Ball Problem: