Geometric Series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r|<1$.

EX: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$

$$1 + \left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^3 + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

Geometric series with $a=1$, $r=-\frac{1}{2}$.

Divergence Test (N-th term test):

If $\sum_{n=1}^{\infty} a_n$ converges, then the terms a_n must tend to zero as $n \to \infty$.

Consequently, if $\lim_{n\to\infty} a_n$ is not zero, then $\sum_{n=1}^{\infty} a_n$ does not converge.

Warning: This is a one way test.

If $a_n \not\to 0$, the series diverges. But if $a_n \to 0$ as $n \to \infty$, we can't conclude anything.

EX: Does $\sum_{n=1}^{\infty} \underbrace{\frac{n}{n+5}}_{a_n}$ converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+\frac{5}{n}} = \frac{1}{1+0} = 1$$ not zero!

Series Diverges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ Terms do $\to 0$, so

Divergence test says nothing.

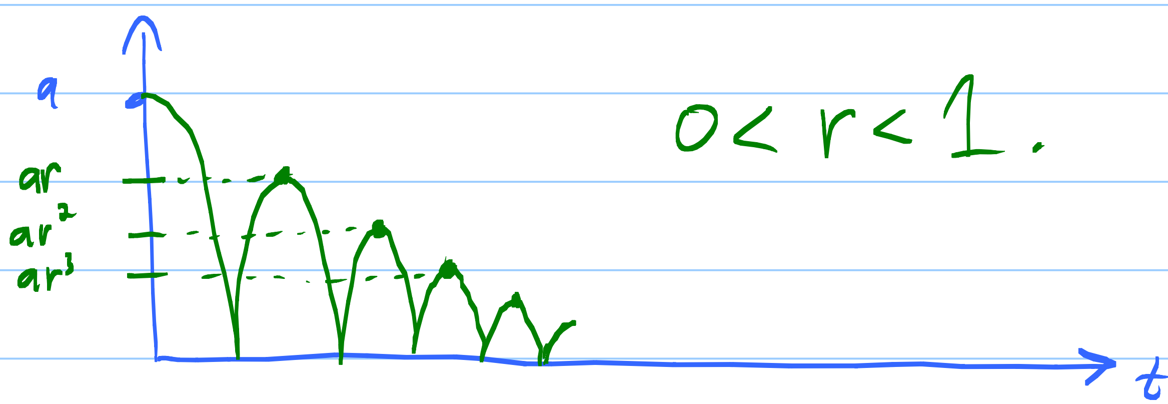
Trick: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

"Telescope" down to $S_n = 1 - \frac{1}{n+1}$.

See $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Bouncing Ball problem:



$0 < r < 1$.

How far does the ball travel?

$$D = a + 2ar + 2ar^2 + 2ar^3 + \cdots$$

$$D + a = 2a + 2ar + 2ar^2 + \cdots = \frac{2a}{1-r}$$

$$D = \frac{2a}{1-r} - a = \frac{a(1+r)}{1-r}$$

How long does it bounce?

← drop from height h

$$h = \frac{1}{2} g t^2 \qquad t = \sqrt{\frac{2h}{g}}$$

$$T = \sqrt{\frac{2a}{g}} + 2\sqrt{\frac{2(ar)}{g}} + 2\sqrt{\frac{2(ar^2)}{g}} + \cdots$$

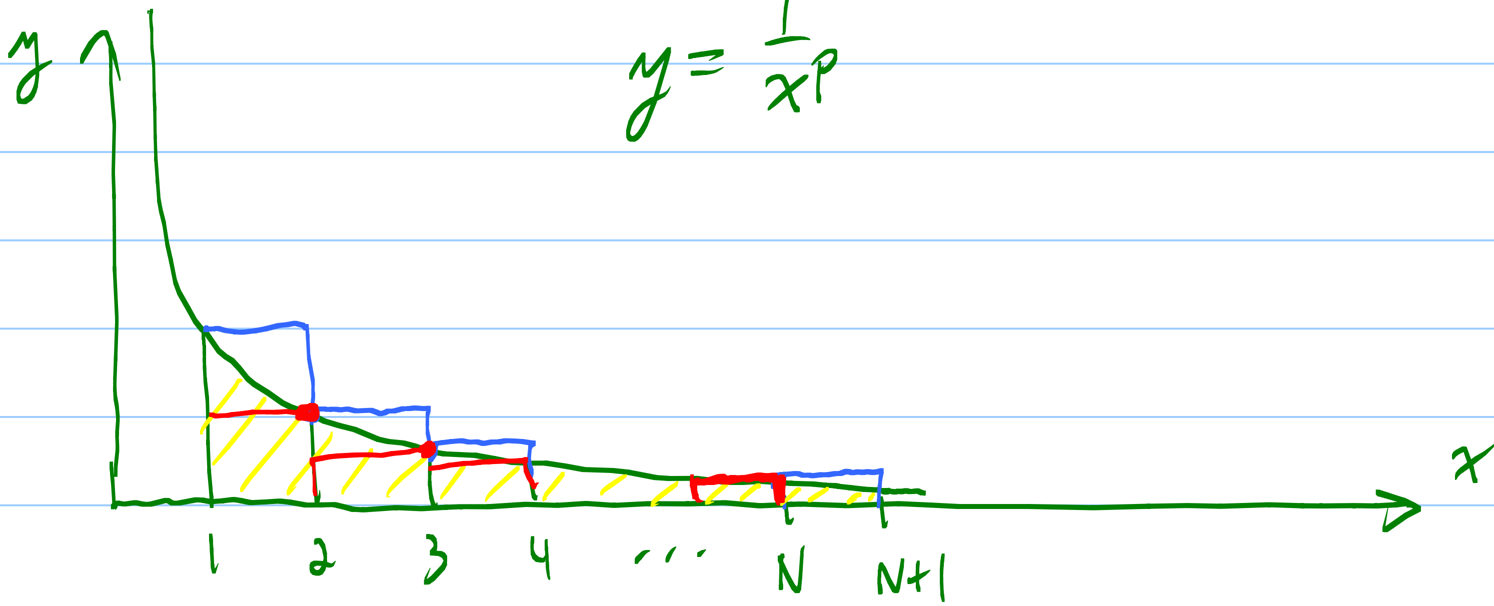
$$T + \sqrt{\frac{2a}{g}} = 2\sqrt{\frac{2a}{g}}\left[1 + \sqrt{r} + (\sqrt{r})^2 + (\sqrt{r})^3 + \cdots\right]$$

$$T + \sqrt{\frac{2a}{g}} = 2\sqrt{\frac{2a}{g}} \cdot \frac{1}{1-\sqrt{r}}$$

Finite T!

__Problem__: For what values $p > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

$$y = \frac{1}{x^p}$$



$$\int_1^{N+1} \frac{1}{x^p}\,dx < \frac{1}{1^p} + \underbrace{\frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{N^p}}_{< \int_1^N \frac{1}{x^p}\,dx} < 1 + \int_1^N \frac{1}{x^p}\,dx$$

Case: $0 < p \le 1$, then $\int_1^\infty \frac{1}{x^p}\,dx = \infty$. Left hand ineq shows $\sum_{n=1}^\infty \frac{1}{n^p} = \infty$ too.

Case: $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx < \infty$ and right hand ineq shows $\sum_{n=1}^\infty \frac{1}{n^p}$ is finite.

($S_N = \sum_{n=1}^N \frac{1}{n^p}$ is an increasing sequence bounded above by $1 + \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} + 1$ So it must converge.)

Integral Test: Suppose $a_n = f(n)$ where f is a continuous function and

1) $f(x) > 0$ for $x > N_0$, and
2) $f(x)$ is decreasing for $x > N_0$.

Then $\sum_{n=N}^\infty a_n$ and $\int_N^\infty f(x)\,dx$ do the same thing. They either both converge or both diverge.

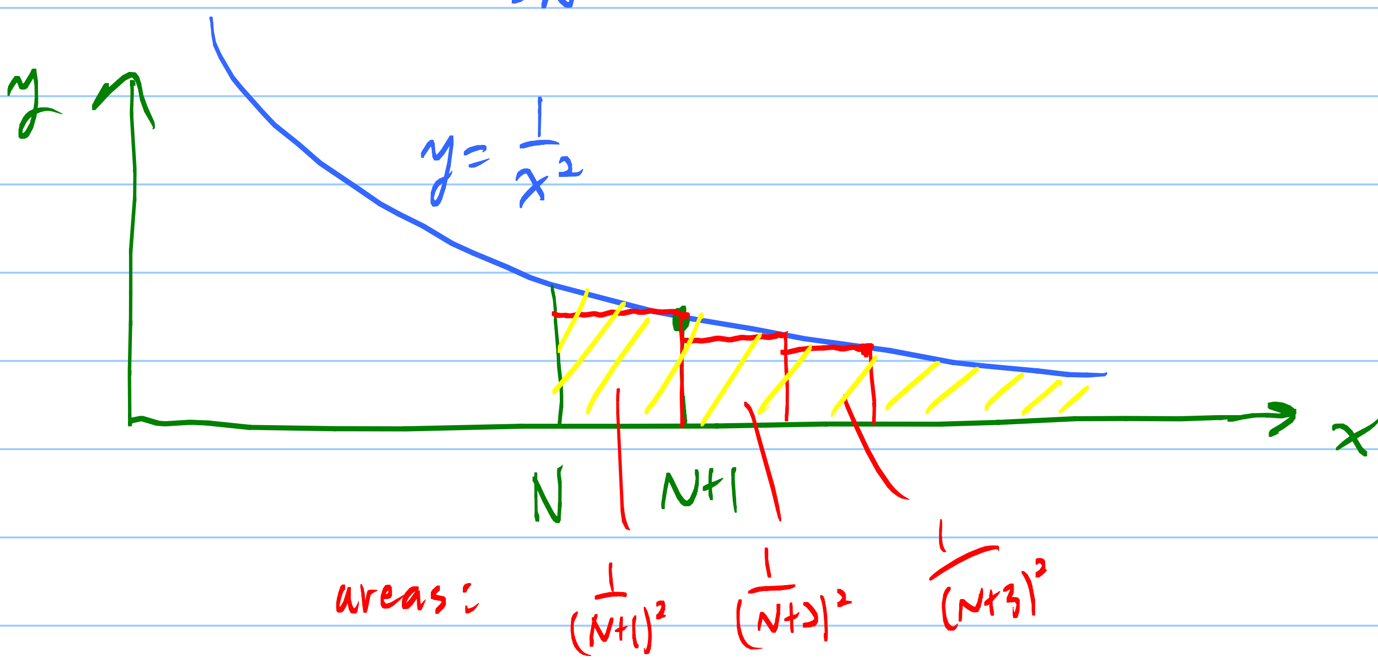
<u>Note</u>: $\underbrace{a_1 + a_2 + \cdots + a_{N-1}}_{\text{just a number.}} + \sum_{n=N}^{\infty} a_n$

So $\sum_{n=1}^{\infty} a_n$ converges exactly when $\sum_{n=N}^{\infty} a_n$ does.

<u>Problem</u>: Famous fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

How many terms do I need to use to get within 10^{-6} of $\pi^2/6$?

$$\frac{\pi^2}{6} = \underbrace{\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{N^2}\right)}_{S_N} + \underbrace{\sum_{n=N+1}^{\infty} \frac{1}{n^2}}_{E = \text{error.}}$$



See $E < \int_N^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_N^{\infty} = 0 - \left(-\frac{1}{N}\right)$

$\left| S_N - \frac{\pi^2}{6} \right| < \frac{1}{N}$ ← Need $N > 10^6$ to make $E < 10^{-6}$!

Homework prob: $\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \ldots$

$$a_{n+1} = \sqrt{1 + a_n}$$ ← recursion formula

Given: Series converges to L. What is L?

By continuity

$$\begin{array}{ccc} a_{n+1} & = & \sqrt{1+a_n} \\ {\scriptstyle n\to\infty}\downarrow & & \downarrow \\ L & = & \sqrt{1+L} \end{array}$$

$$L^2 = 1 + L$$

$$L^2 - L - 1 = 0$$

$$L = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$ ← L not negative.

So $L = \frac{1+\sqrt{5}}{2}$. Golden ratio!