p. 560: 98. $\sqrt{1}, \sqrt{1+\sqrt{1}}, \ldots$

$$a_{n+1} = \sqrt{1 + a_n} \qquad a_0 = 0.$$

$$\downarrow \qquad \downarrow$$

$$L = \sqrt{1+L}$$

$L^2 = 1 + L$ $\qquad$ $L =$ positive root $\frac{1+\sqrt{5}}{2}$.

How to see that a_n do converge.

Fact 1: If $0 < a_n < \frac{1+\sqrt{5}}{2}$, then a_{n+1} falls in the same range.

Why: Suppose $0 < a_n < \frac{1+\sqrt{5}}{2}$

Then $0+1 < 1+a_n < \frac{3+\sqrt{5}}{2}$

$$\sqrt{1} < \underbrace{\sqrt{1+a_n}}_{a_n} < \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}!$$

Fact 2: $a_{n+1} > a_n > a_{n-1} > \ldots$

$0 < a_n < \frac{1+\sqrt{5}}{2}$ $\leftarrow$ Fact 1 says a_n stuck in here

$$\sqrt{1+a_n} \overset{?}{>} a_n$$

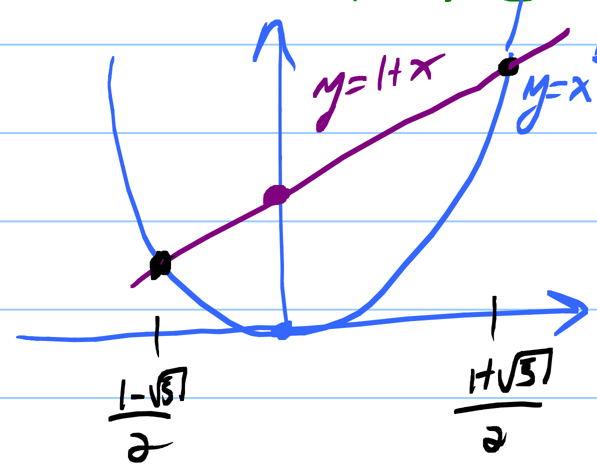
$$1 + a_n \overset{?}{>} a_n^2$$

yes!

$$\|$$

$$a_{n+1}^2 > a_n^2$$

So $a_{n+1} > a_n$. ✓



Conclude that $\{a_n\}_{n=0}^{\infty}$ is an increasing sequence bounded above by $\frac{1+\sqrt{5}}{2}$. So it has to converge.

From above, $\lim = \frac{1+\sqrt{5}}{2}$.

Divergence Test (N-th term test):

If $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

(Hence, if $a_n \not\to 0$, then $\sum a_n$ diverges.)

Why:

$$S_N = a_1 + a_2 + \cdots + a_{N-1} + a_N$$

$$S_{N-1} = a_1 + a_2 + \cdots + a_{N-1}$$

$$S_N - S_{N-1} = a_N \qquad \text{Let } N \to \infty:$$

$$\begin{array}{ccccc} S_N & - & S_{N-1} & = & a_N \\ \downarrow & & \downarrow & & \downarrow \\ L & - & L & = & 0 \end{array}$$

EX: $.99\bar{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

$$= \frac{9}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right] = \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = 1$$

10.4 Comparison Tests:

Basic Convergence Test: Suppose $a_n \geq 0$, $b_n \geq 0$ and $a_n \leq b_n$ for $n > N$.

If $\sum_1^{\infty} b_n$ converges, then so does $\sum_1^{\infty} a_n$.

If $\sum_1^\infty a_n$ diverges, the so does $\sum_1^\infty b_n$.

EX: Does $\sum_{n=1}^\infty \frac{6n}{n^3+1}$ converge?

$$\underbrace{\frac{6n}{n^3+1}}_{a_n} < \frac{6n}{n^3} = \underbrace{\frac{6}{n^2}}_{b_n}$$

b_n p-series $\sum \frac{1}{n^p}$ with $p=2$.

Conv. $p>1$.

Div. $0<p\le 1$.

$\sum_1^\infty \frac{6}{n^2}$ conv. So does $\sum_1^\infty \frac{6n}{n^3+1}$.

EX: Does $\sum_{n=0}^\infty \frac{1}{n!}$ converge? (Yes, to e.)

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdots(n-1)n} < \frac{1}{(n-1)n} = \underbrace{\frac{1}{n-1} - \frac{1}{n}}_{\text{telescoping series. Converges.}}$$

compare

Conclude $\sum \frac{1}{n!}$ conv. too.

OR $$\frac{1}{1\cdot 2\cdot 3\cdots(n-1)\cdot n} < \frac{1}{1\cdot\underbrace{2\cdot 2\cdot 2\cdots 2}_{n-1\ 2\text{'s}}} = \frac{1}{2^{n-1}}$$

Compare $\sum \frac{1}{n!}$ to geometric series $\sum \frac{1}{2^{n-1}}$.

EX: Does $\sum_{n=1}^\infty \frac{\ln n}{n}$ converge?

$\ln n > 1$ for $n \ge 3$. so $\frac{\ln n}{n} > \frac{1}{n}$ for $n\ge 3$.

But $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So $\sum \frac{\ln n}{n}$ diverges too.

Limit Comparison Test: Suppose $a_n > 0$, $b_n > 0$.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$,

then $\sum a_n$ and $\sum b_n$ do the same thing.

Fancy Part II:

A) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then, (Think $a_n << b_n$.)

if $\sum b_n$ converges, so does $\sum a_n$.

B) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then, (Think $a_n >> b_n$)

if $\sum b_n$ diverges, so does $\sum a_n$.

EX: Does $\sum_{n=1}^{\infty} \frac{\sqrt{n^7} - 7}{n^4 + n - 1}$ converge?

$$\underbrace{\frac{\sqrt{n^7} - 7}{n^4 + n - 1}}_{a_n} \sim \frac{\sqrt{n^7}}{n^4} = \underbrace{\frac{1}{n^{7/2}}}_{b_n} \leftarrow \text{p-series } p = \tfrac{7}{2}. \text{ conv.!}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^7} - 7}{n^4 + n - 1}\right)}{\left(1/n^{7/2}\right)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\left(1-\frac{7}{\sqrt{n}}\right)n^{7/2}}{n^4\left(1+\frac{1}{n^3}-\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{\left(1-\frac{7}{\sqrt{n}}\right)}{1+\frac{1}{n^3}-\frac{1}{n^4}}$$

$$= 1 \leftarrow L, \quad 0 < L < \infty.$$

Series converges because the p-series does.

EX: Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ conv. ?

Idea: Compare to a p-series $\sum \frac{1}{n^p}$.

$$\lim_{n\to\infty} \frac{\left(\frac{\ln n}{n^2}\right) \leftarrow a_n}{\left(\frac{1}{n^p}\right) \leftarrow b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{2-p}} = 0 \text{ if } \underbrace{2-p>0}_{p<2}.$$

p-series converges if $1 < p$. Aha! Pick any p between 1 and 2, How about $p = \frac{3}{2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \quad \text{if} \quad b_n = \frac{1}{n^{3/2}}$$

Fancy Part II shows series converges.