Don't forget to do the online evaluation for Cuiyu He.

EX: Does $1^{-2} + 2^{-3/2} + 3^{-4/3} + 4^{-5/4} + \cdots$ converge?

$$a_n = n^{-(n+1)/n} = \frac{1}{n^{1+\frac{1}{n}}}$$

$$= \frac{1}{n\, n^{1/n}} \sim \frac{1}{n} = b_n$$

remember $\sqrt[n]{n} \to 1$ as $n \to \infty$

Limit Comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1.$

Given series $\sum a_n$ and $\sum \frac{1}{n}$ do same thing. $\sum \frac{1}{n}$ diverges, so $\sum a_n$ diverges too.

10.5 Ratio Test and Root Test:

Ratio Test: $a_n > 0.$

Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is equal to L.

Then the series $\sum_{n=1}^{\infty} a_n$

A) Converges if $0 \le L < 1$,

B) Diverges if $L > 1$, and

c) the test fails if $L = 1$.

Why: Case A: Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$.

Pick an r between L and 1. Because $L < r$, there is an N such that

$$\frac{a_{n+1}}{a_n} < r \quad \text{when } n \geq N.$$

$n = N$: $\quad \frac{a_{N+1}}{a_N} < r \qquad \underbrace{a_{N+1} < a_N r}$

$n = N+1$: $\quad \frac{a_{N+2}}{a_{N+1}} < r \qquad \underbrace{a_{N+2} < a_{N+1} r < a_N r^2}$

$n = N+2$: $\qquad a_{N+3} < a_{N+2} r < a_N r^3$

$\vdots \qquad\qquad \vdots$

$n = N+99 \qquad a_{N+100} < a_N r^{100}$

$\vdots \qquad\qquad \vdots$

See
$$\begin{array}{ccccccccc} a_1 + a_2 + a_3 + \cdots & + & a_N & + & a_{N+1} & + & a_{N+2} & + & \cdots \\ \| & & \| & & \wedge & & \wedge & & \wedge \\ < \underbrace{a_1 + a_2 + \cdots + a_{N-1}}_{B} & + & \multicolumn{7}{c}{\underbrace{a_N + a_N r + a_N r^2 + a_N r^3 + \cdots}_{\text{geom series, } r<1}} \end{array}$$

$$B + \frac{a_N}{1-r} \leftarrow \text{finite!}$$

So $\sum a_n$ is finite, i.e., it converges.

<u>Case B</u>: $\frac{a_{n+1}}{a_n} \to L > 1$

Then $\frac{a_{n+1}}{a_n} > 1$ for $n \geq N$.

See $\cdots > a_{n+3} > a_{n+2} > a_{n+1} > a_n$

Ouch! The terms don't go to zero!

So series diverges.

<u>Case C</u>: $\sum \frac{1}{n}$ $L = 1$ $\leftarrow$ diverges!

$\sum \frac{1}{n^2}$ $L = 1$ $\leftarrow$ converges!

Test fails.

<u>EX</u>: $\sum_{n=0}^{\infty} \frac{10^n}{n!}$

(Notational conventions $x^0 = 1$ even when $x=0$, $0! = 1$ too.)

$a_n = \frac{10^n}{n!}$ $a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{10^{n+1}}{1\cdot 2\cdot 3 \cdots n(n+1)}\right)}{\left(\frac{10^n}{1\cdot 2\cdot 3\cdots n}\right)} = \frac{10}{n+1} \xrightarrow[\text{as } n\to\infty]{} 0 = L$$

Since $L < 1$, Ratio Test says this series converges.

In fact, $\sum_0^{\infty} \frac{10^n}{n!} = e^{10}$. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Prob: For what values of b does

$$\sum_{n=1}^{\infty} \frac{b^n}{n\,2^n} \quad \text{converge?}$$

$$a_n = \frac{b^n}{n 2^n} \qquad a_{n+1} = \frac{b^{n+1}}{(n+1)\,2^{(n+1)}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{b^{n+1}}{(n+1)\,2^{n+1}}\right)}{\left(\frac{b^n}{n\,2^n}\right)} = \frac{b}{2} \cdot \frac{n}{n+1} \to \frac{b}{2}$$

as $n \to \infty$.

$$\frac{b}{2} = L \begin{array}{ll} <1 & \text{conv.} \\ >1 & \text{div.} \end{array} \qquad \text{So} \quad \begin{array}{ll} b<2 & \text{conv.} \\ b>2 & \text{div.} \end{array}$$

$b=2$. Ratio Test Fails. But in this case

$$a_n = \frac{2^n}{n\,2^n} = \frac{1}{n} \leftarrow \text{famous divergent series.}$$

EX: $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{(n+1)!}{[2(n+1)]!}\right)}{\left(\frac{n!}{(2n)!}\right)} = \frac{\left(\frac{1\cdot 2\cdot 3\cdots n\cdot(n+1)}{1\cdot 2\cdot 3\cdots 2n\cdot(2n+1)(2n+2)}\right)}{\left(\frac{1\cdot 2\cdots\cdots n}{1\cdot 2\cdots\cdots 2n}\right)}$$

$$= \frac{n+1}{(2n+1)(2n+2)} \longrightarrow \underset{\nwarrow L}{0} \quad \text{as} \quad n \to \infty$$

$L < 1$. So Series converges.

Root Test: $a_n > 0$.

Suppose $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and is equal L.

$\sum_{n=1}^{\infty} a_n$

A) Converges if $0 \le L < 1$

B) Diverges if $L > 1$

c) test fails if $L = 1$.

Why: Compare to geometric series. See p. 584.

EX: $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ $\qquad a_n = \frac{3^n}{n^3}$

$$\sqrt[n]{a_n} = \left(\frac{3^n}{n^3}\right)^{1/n} = \frac{3}{n^{3/n}} = \frac{3}{(n^{1/n})^3}$$

$$\xrightarrow[\text{as } n\to\infty]{} \frac{3}{1^3} = 3 = L. \qquad L > 1.$$

Series Diverges.

EX: $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$ $\qquad a_n = \frac{n+1}{2^n}$

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{n+1}}{\sqrt[n]{2^n}} = \frac{\sqrt[n]{n(1+\frac{1}{n})}}{2} = \frac{\sqrt[n]{n}\,\sqrt[n]{1+\frac{1}{n}}}{2}$$

Hmmm. $1 < 1+\frac{1}{n} < 2$

$$1^{1/n} < \left(1+\frac{1}{n}\right)^{1/n} < 2^{1/n}$$

$$\begin{array}{ccc} \| & \uparrow & \downarrow \\ 1 & & 1 \end{array}$$

sandwich thm says $\left(1+\frac{1}{n}\right)^{1/n} \to 1$ as $n \to \infty$.

So $\sqrt[n]{a_n} = \dfrac{\sqrt[n]{n}\,\sqrt[n]{1+1/n}}{2} \to \dfrac{1}{2} = L$ as $n \to \infty$. $L < 1$. So series converges.

Famous Series: Leibnitz's Series for $\frac{\pi}{4}$.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

<u>How</u>: Crazy wild thinking:

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

Let $r = -x^2$

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1-(-x^2)} = \frac{1}{x^2+1}$$

Integrate $\int_0^1 dx$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \operatorname{Tan}^{-1} x \Big|_0^1 = \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

Wow! Can we really do that?

Write that geometric series with partial sum plus error.