Def^n: $\sum_1^\infty u_n$ is <u>absolutely</u> convergent if $\sum_1^\infty |u_n|$ is finite.

Fact: Absolutely convergent series converge!

why: $-|u_n| \le u_n \le |u_n|$ ← add $|u_n|$

$$0 \le u_n + |u_n| \le 2|u_n|$$

$\sum_{n=1}^\infty u_n + |u_n|$ converges too by comparison thm.

$2\sum |u_n|$ converges

$$\underbrace{\sum_{n=1}^N \left(u_n + |u_n|\right)}_{\downarrow \; L_1} = \sum_{n=1}^N u_n + \sum_{n=1}^N |u_n|$$

must converge to

$$L_1 = (L_1 - L_2) \quad L_2$$

Classic Examples: $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$ converges by alt. series test, but it does not converge absolutely. It is <u>conditionally</u> convergent.

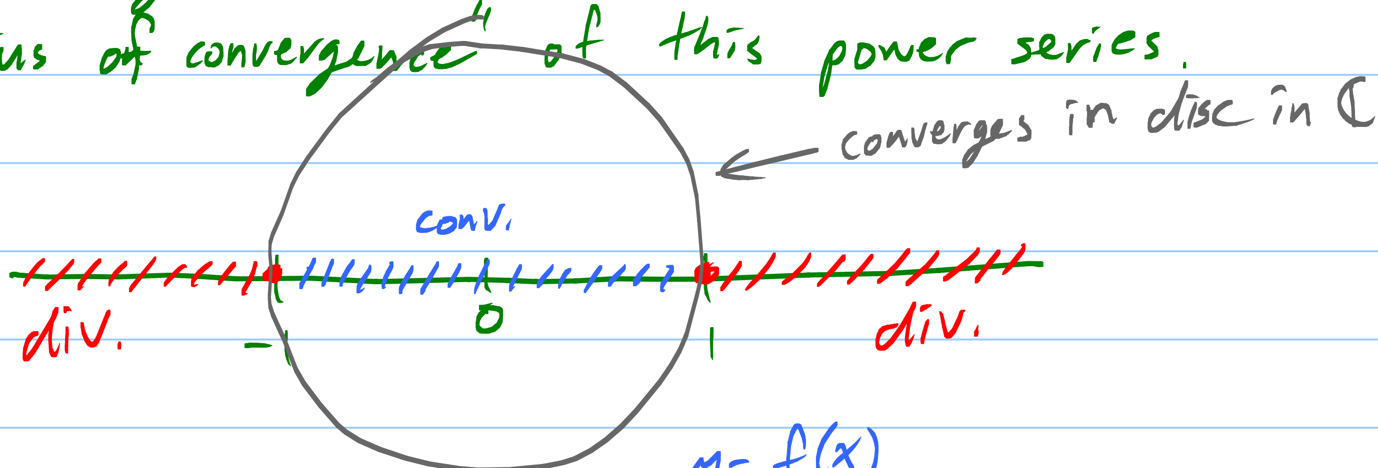
Freaky fact: Can rearrange absolutely conv. series and they still converge to the same thing. Not so for conditionally convergent series. Can rearrange them and make them converge to any real number!

10.7 Power Series: $\sum_{n=0}^{\infty} u_n$ where

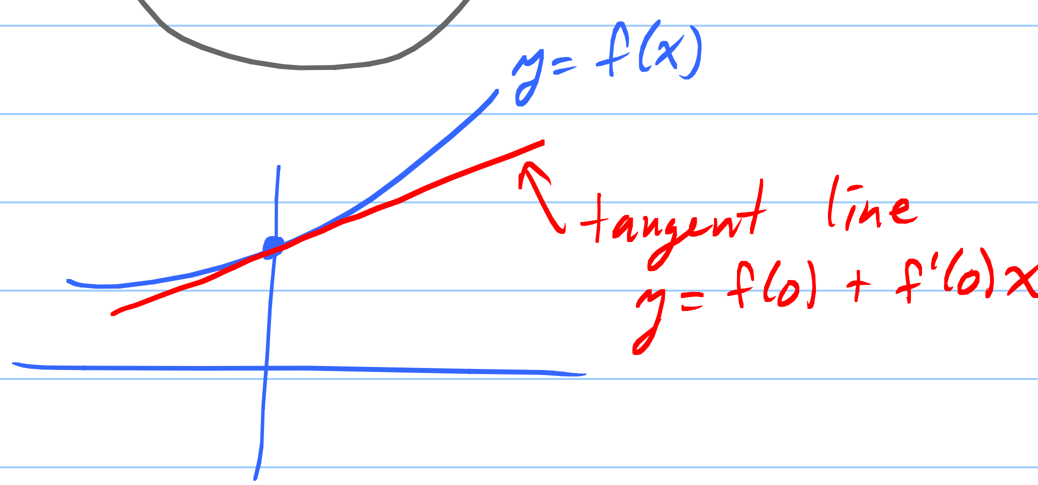
$u_n = a_n x^n$.

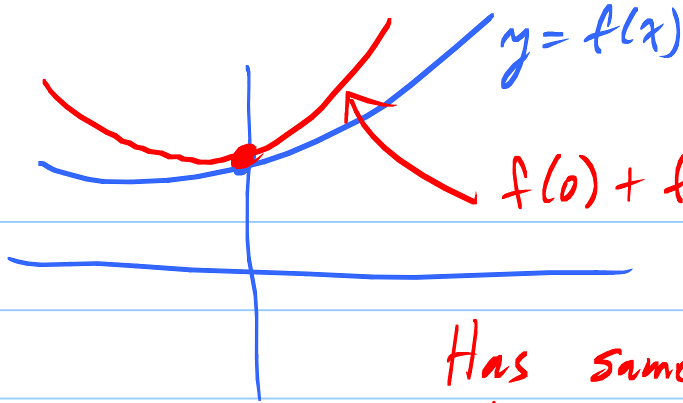
EX: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

is a power series and it converges if $|x| < 1$. It diverges if $|x| > 1$. Word: 1 is the "radius of convergence" of this power series.



Big idea:





Has same value, same slope, and second deriv. at $x=0$ as $f(x)$.

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$P_3'(x) = a_1 + 2a_2 x + 3a_3 x^2$$
$$P_3''(x) = 2a_2 + 3\cdot 2\, a_3 x$$
$$P_3'''(x) = 3\cdot 2\, a_3$$

want

$$P_3(0) = a_0 = f(0)$$
$$P_3'(0) = a_1 = f'(0)$$
$$P_3''(0) = 2a_2 = f''(0)$$
$$P_3'''(0) = 3\cdot 2 a_3 = f'''(0)$$

Need $a_3 = \frac{f'''(0)}{3\cdot 2} = \frac{f'''(0)}{3!}$.

Can do this for n-th degree poly

$$P_N(x) = a_0 + a_1 x + \cdots + a_N x^N.$$

Match values at $x=0$ up to order N.
Get "Taylor's Formula" $a_n = \frac{f^{(n)}(0)}{n!}$

[Note: $0! = 1$. $f^{(0)}(0) = f(0)$ by convention.]

Problem: Find Taylor Series for $\sin x$.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0 \leftarrow$ repeats.

$\vdots \qquad \vdots$

$$\text{Taylor Series} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 0 + \frac{1}{1!}x + 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 - 0x^6 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

EXAMPLES: $\displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{n}{2^n} x^n}_{u_n}$

Find the Radius of conv. Use Ratio Test.

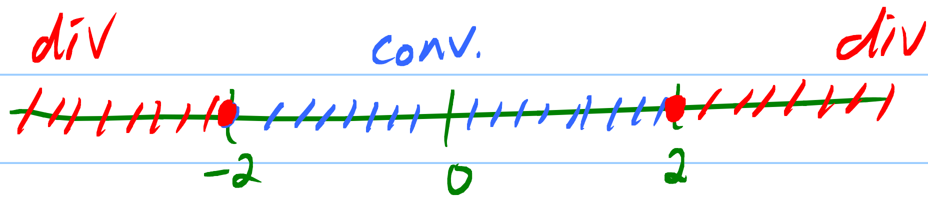
New improved Ratio Test: $L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$

IF $L<1$, then $\sum |u_n|$ converges, and $\sum u_n$ converges absolutely. If $L>1$, it diverges.

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n}\right| = \frac{n+1}{n} \cdot \frac{1}{2} |x| \to \frac{1}{2}|x| \begin{array}{ll} <1 & \text{Conv.} \\ >1 & \text{div.} \end{array}$$

See $|x| < 2$ gives absolute convergence.

$|x| > 2$ gives divergence.



$R = 2$

At endpoints $\sum \underbrace{\frac{n}{2^n}(\pm 2)^n}_{\pm n}$ ← diverges because terms do not $\to 0$.

EX: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio Test shows converges for all x. $R = \infty$.