591.49: $L = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}_{S_4} + \frac{1}{5} - \dots$

$$|L - S_4| < \underbrace{a_5}_{\frac{1}{5}}$$

Method to see that a_n decreasing

$a_n = \frac{1}{n^2+1}$ easy.

EX: $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ $\qquad a_n = \frac{\ln n}{n}$

1) $a_n = \frac{\ln n}{n} > 0$ ✓

2) a_n decreasing? Trick: $f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{x \cdot \frac{1}{x} - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$1 - \ln x = 0$ $\qquad 1 - \ln x > 0 \quad 0 < x < e$

$1 = \ln x$ $\qquad 1 - \ln x < 0 \quad x > e$

$e = x$

$f(x)$ is decreasing from e on.

So a_n decreasing from $n=3$ on. ✓

3) $a_n \to 0$? $\lim_{x\to\infty} \frac{\ln x}{x} = 0.$ ✓

Alt. Series Test: $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$ converges.

<u>Fact</u>: Power series converge <u>absolutely</u> inside their radius of convergence R (i.e., inside $|x| < R$). OK to differentiate or integrate term by term and the resulting power series has the same R. of C.

<u>EX</u>: $(*)$ $\qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, $R = 1$

$\frac{d}{dx}(*)$ $\qquad \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots$, $R = 1$

$\int (*)$ $\qquad -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$, $R = 1$

<u>EX</u>: $f(x) = \ln(1+x)$. Find the Taylor Series for $f(x)$ centered at $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\ln(1+x)$	$\ln 1 = 0$	$0/0! = 0$
1	$1/(1+x)$	1	$1/1! = 1$
2	$-1/(1+x)^2$	-1	$-1/2! = -1/2$
3	$\frac{(-1)(-2)}{(1+x)^3}$	$(-1)(-2)$	$\frac{(-1)(-2)}{3!} = \frac{1}{3}$
4	$\frac{(-1)(-2)(-3)}{(1+x)^4}$	$(-1)(-2)(-3)$	$\frac{(-1)(-2)(-3)}{4!} = -\frac{1}{4}$
$\vdots$			$\vdots$
n		$f^{(n)}(0) = (-1)^{n+1}(n-1)!$	$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}$

N-th Taylor Polynomial:

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} x^n$$

<u>Taylor Series centered at $x=a$:</u>

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

matches value and derivatives up to order N at $x=a$ with $f(x)$. The case $a=0$ is called the Maclaurin Series for f.

<u>Prob</u>: Find Taylor Series for $f(x) = \frac{1}{x}$ about $x=2$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$\frac{f^{(n)}(2)}{n!}$
0	$1/x$	$1/2$	$(1/2)/0! = 1/2$
1	$-1/x^2$	$-\frac{1}{2^2}$	$\frac{-1/2^2}{1!} = -\frac{1}{2^2}$
2	$\frac{(-1)(-2)}{x^3}$	$\frac{(-1)(-2)}{2^3}$	$\frac{1}{2^3}$
3	$\frac{(-1)(-2)(-3)}{x^4}$	$\frac{(-1)(-2)(-3)}{2^4}$	$\frac{(-1)^3}{2^4}$
$\vdots$			
n			$\frac{(-1)^n}{2^{n+1}}$

Taylor Series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$

Huge Fact: Taylor's Theorem with Remainder:

$$f(x) = P_N(x) + R_N(x) \quad \text{where}$$

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

where c is some point between a and x.
[Note: c can depend on x.]

Application: Power Series for $\sin x$ converges to $\sin x$. $a=0$: $\sin x \stackrel{?}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

If $f(x) = \sin x$, then

$$f^{(N+1)}(x) = \pm \sin x \quad \text{or} \quad \pm \cos x$$

$$\left| f^{(N+1)}(c) \right| = |\sin c| \quad \text{or} \quad |\cos c| \leq 1$$

$$|R_N(x)| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right| \leq \frac{|x|^{N+1}}{(N+1)!} \to 0$$

as $N \to \infty$ for any fixed x.

ERROR ON $[0, \pi]$: $|R_N(x)| \le \frac{\pi^{N+1}}{(N+1)!}$

Classic examples:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad R = 1.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad R = \infty.$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \frac{|x|}{n+1} \to \underset{\uparrow\, L}{0}$$

$L < 1$. So series converges for all x.
$R = \infty$.

$\sum_{n=0}^{\infty} n!\, x^n$. Ratio Test: $\left|\frac{u_{n+1}}{u_n}\right| = (n+1)|x|$

$\to \infty$ if $x \neq 0$. Series diverges if $x \neq 0$. Only converges at $x = 0$. $R = 0$.