$$x^2 = 0 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3 + 0\cdot x^4 + \cdots$$

$$x^2 \cos x^2 = x^2\left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots\right)$$

---

e^x : The function that satisfies

$$\frac{dy}{dx} = y \quad \text{with} \quad y(0) = 1.$$

Assume solution to the ODE has a convergent power series expansion : $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Want $y'(x) = y(x)$

want

$$0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$$

Must have

$a_1 = a_0 \leftarrow a_0 = y(0) = 1$, So $a_1 = 1$

$2a_2 = a_1 \leftarrow a_2 = \frac{1}{2}a_1 = \frac{1}{2} = \frac{1}{2!}$

$3a_3 = a_2 \leftarrow a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2} = \frac{1}{3!}$

$4a_4 = a_3 \leftarrow a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2} = \frac{1}{4!}$

$\vdots$

$n a_n = a_{n-1} \leftarrow a_n = \frac{1}{n}a_{n-1} = \frac{1}{n}\cdot\frac{1}{(n-1)!} = \frac{1}{n!}$

$\vdots$

See $y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Use ratio test to see this series has R. of C. $= \infty$.

Fun thing to do: Airy's Eqn $y'' - x^2 y = 0$

$$\begin{cases} y(0) = a_0 \\ y'(0) = a_1 \end{cases}$$

Plug in $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into ODE. Determine all the a_n's. Get

$$y(x) = a_0 \big(\text{power series} \big) + a_1 \big(\text{power series} \big)$$

Airy's functions

Euler could not resist this:

$$e^{ix} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + ix - \frac{1}{2!}x^2 - \frac{i}{3!}x^3 + \frac{x^4}{4!} + \cdots$$

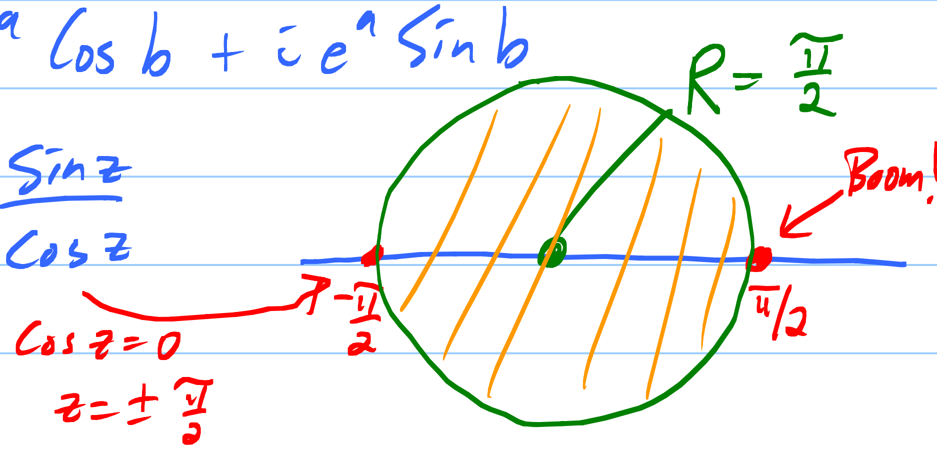
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + i \sin x$$

$$e^{a+bi} = e^a e^{bi} = e^a \cos b + i e^a \sin b$$

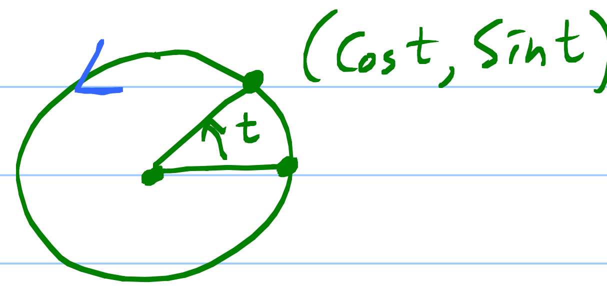
In $\mathbb{C}$

$$\tan z = \frac{\sin z}{\cos z}$$



Chapter II: Curves in $\mathbb{R}^2$.

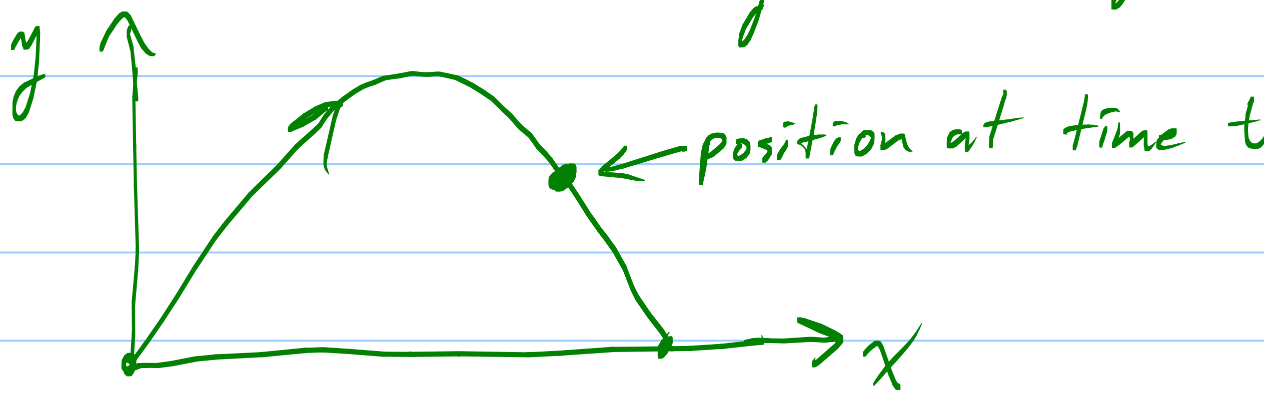
Parametric curves:



$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad 0 \le t \le 2\pi$$

EX: Throw a ball: $x = v_1 \cdot t$

$$y = v_2 t - \tfrac{1}{2} g t^2$$



Problem: Find the tangent line to the circle above at $t = \pi/6$.

$$x = \cos t \qquad \frac{dy}{dx} = ?$$
$$y = \sin t$$

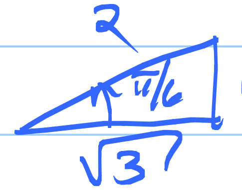
$$\frac{dy}{dx} = \underbrace{\frac{dy}{dt} \cdot \frac{dt}{dx}}_{\text{Chain Rule}} = \frac{dy}{dt} \cdot \frac{1}{\left(\frac{dx}{dt}\right)} = \boxed{\frac{dy/dt}{dx/dt}}$$

inverse function thm

So $\dfrac{dy}{dx} = \dfrac{\cos t}{-\sin t} = -\cot t$

At $t = \frac{\pi}{6}$: $\dfrac{dy}{dx} = -\cot \frac{\pi}{6}$

$= -\sqrt{3}$



Tangent line: $\begin{cases} x = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ y = \sin \frac{\pi}{6} = \frac{1}{2} \end{cases}$

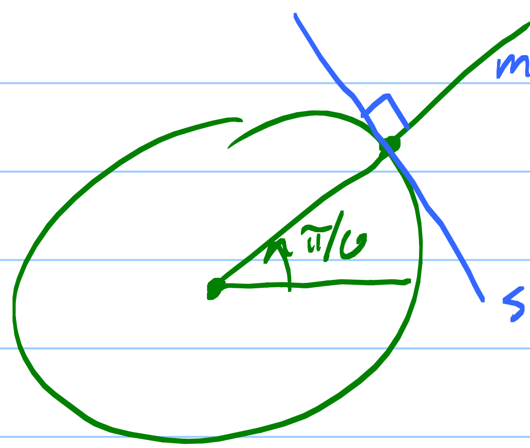
$\dfrac{y - \frac{1}{2}}{x - \frac{\sqrt{3}}{2}} = -\sqrt{3}$

$$\boxed{y = \frac{1}{2} - \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)}$$

Tangent line at $t = \frac{\pi}{6}$ point.

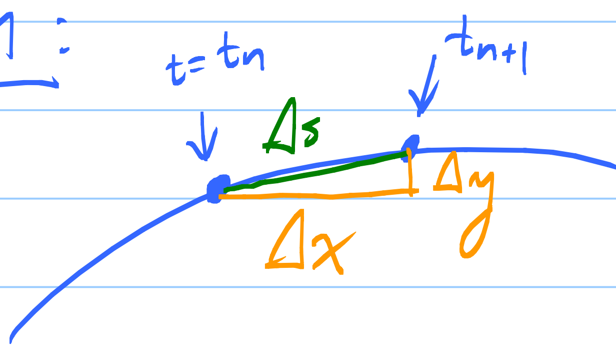
Geometry:

slope $= \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$



slope $= -\frac{1}{m} = -\sqrt{3}$

Arc length:



$x = f(t)$

$y = g(t)$

$\dfrac{\Delta x}{\Delta t} \approx f'(t_n)$ $\qquad$ $\dfrac{\Delta y}{\Delta t} \approx g'(t_n)$

where $\Delta t = t_{n+1} - t_n$.

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(f'(t_n)\Delta t)^2 + (g'(t_n)\Delta t)^2}$$

$$= \sqrt{f'(t_n)^2 + g'(t_n)^2}\,\Delta t$$

Total arc length $s \approx \sum \Delta s$ = Riemann Sum

Let Δt's $\to 0$. See

$$s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt$$

How to remember:

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Think: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{dx^2 + dy^2} = ds$

$$s = \int ds$$

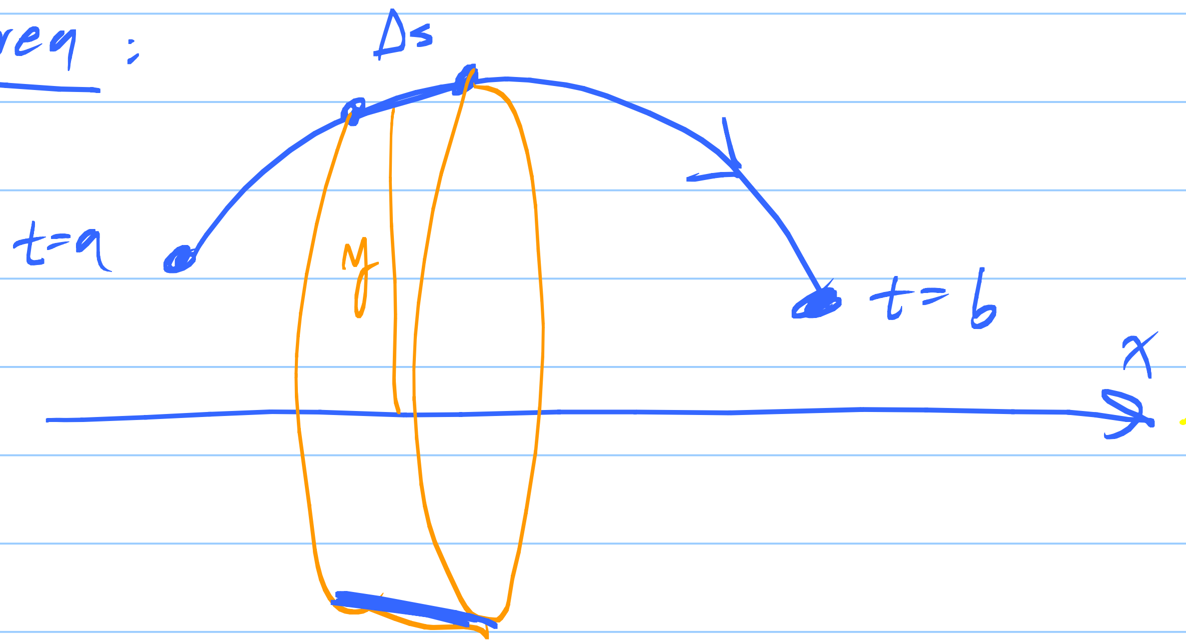
Circle: $f(t) = \cos t$ $\qquad f'(t) = -\sin t$

$g(t) = \sin t$ $\qquad g'(t) = \cos t$

$$S = \int_0^{2\pi} \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1}}\, dt = \int_0^{2\pi} 1\, dt$$

$$= t \Big]_0^{2\pi} = 2\pi \quad \checkmark$$

Surface Area :



$$A = \int_a^b 2\pi y\, ds = \int_a^b 2\pi\, g(t) \sqrt{f'(t)^2 + g'(t)^2}\, dt$$