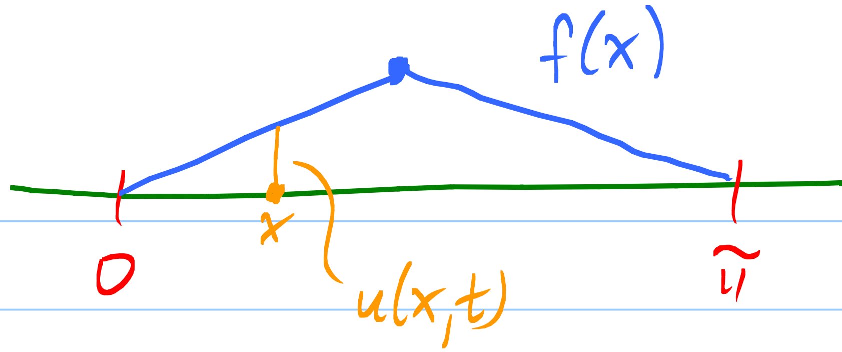


$$\frac{\partial^2 u}{\partial x^2} = c\,\frac{\partial^2 u}{\partial t^2} \qquad (\text{Take } c = 1)$$

Wave Equation
PDE

I.C. $\begin{cases} u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$ B.C. $\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

Jacob Bernouli : $u(x,t) = \varphi(x-t) + \psi(x+t)$

Johann : Try $u(x,t) = X(x)\,T(t)$

Plug into PDE: $\dfrac{\partial^2}{\partial x^2}(X\,T) = \dfrac{\partial^2}{\partial t^2}(X\,T)$

$$X''\,T = X\,T''$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

Must be constant.
(Fix $t=0$. See const. in x.
Fix x. See it is same const. in t.)

From one PDE, get two ODE:

$$X'' - \lambda X = 0 \quad \Big| \quad T'' - \lambda T = 0$$

Hardest thing: Initial Conditions. Save.

Boundary Conditions: Need

$$u(0,t) = X(0)\,T(t) = 0$$
$$u(\pi,t) = X(\pi)\,\underbrace{T(t)} = 0$$

want

need

$X(0) = 0$, $X(\pi) = 0$

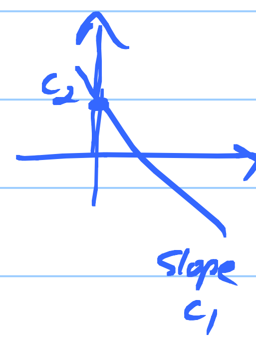
Don't want $T(t) \equiv 0$. (Know the zero sol^n.)

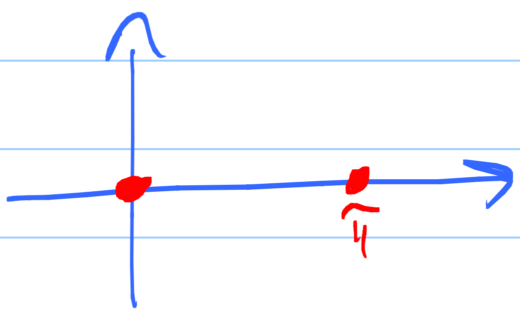
Sturm-Liouville Prob:
$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = 0, \quad X(\pi) = 0 \end{cases}$$

Case $\lambda = 0$: $X'' = 0$

$$X' = c_1$$
$$X = c_1 x + c_2$$

line





$X(x) \equiv 0$

Darn! Only get zero sol^n.

Case $\lambda > 0$: $X'' - \lambda X = 0$

Try $X(x) = e^{rx}$: Need $r^2 - \lambda = 0$

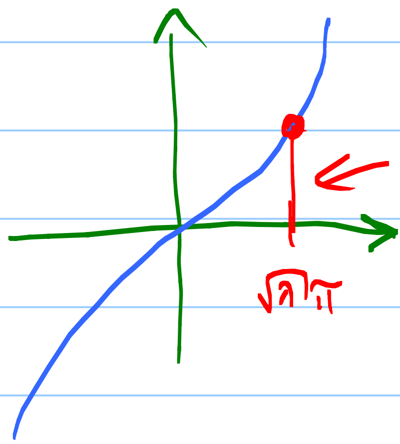
$r = \pm\sqrt{\lambda}$. Gen^l Sol^n $X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

B.C.: Need $\underline{X}(0) = c_1 + c_2 \overset{\text{want}}{=} 0$ $\boxed{c_2 = -c_1}$

$$\underline{X}(\pi) = c_1 e^{\sqrt{\lambda}\pi} - c_1 e^{-\sqrt{\lambda}\pi} \overset{\text{want}}{=} 0$$

$$\underline{X}(\pi) = 2c_1\left(\frac{e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}}{2}\right)$$

$$= 2c_1 \sinh\sqrt{\lambda}\pi \overset{\text{want}}{=} 0$$



$\sinh\sqrt{\lambda}\pi \neq 0$ ouch!

Must take $c_1 = 0$.

$c_2 = -c_1 = 0$ too!

Darn! Zero solⁿ.

Case $\lambda < 0$: Write $\lambda = -\mu^2$ $(\mu > 0.\ \mu \neq 0)$.

$$\underline{X}'' - \lambda \underline{X} = 0$$

$$\underline{X}'' + \mu^2 \underline{X} = 0$$

$r^2 + \mu^2 = 0$ $\quad r = \pm \mu i$ $\quad$ Complex roots!

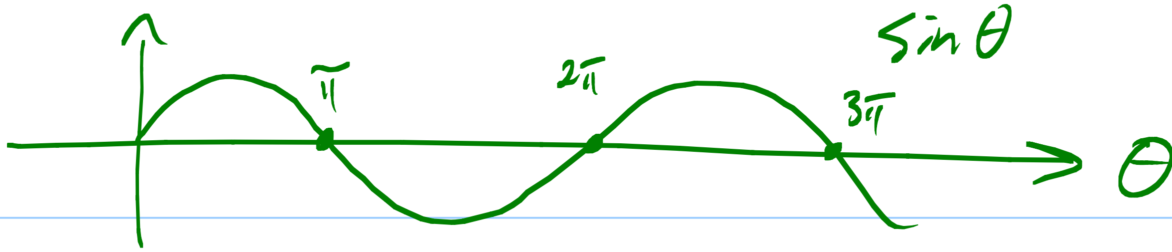
Complex Solⁿ $\quad e^{\mu i x} = \underbrace{\cos \mu x}_{} + i\, \underbrace{\sin \mu x}_{}$

Two real solⁿs!

Genl Solⁿ: $\underline{X}(x) = c_1 \sin \mu x + c_2 \cos \mu x$

B.C. $\quad \underline{X}(0) = c_1 \cdot 0 + c_2 \cdot 1 = c_2 \overset{\text{want}}{=} 0$ $\quad \boxed{c_2 = 0}$

$$\underline{X}(\pi) = c_1 \sin \mu\pi \overset{\text{want}}{=} 0$$



Don't want $c_1 = 0$. Need $\sin \mu\pi = 0$.

μ that work: $\mu\pi = n\pi$ $\quad n = 1, 2, 3, \dots$

$\boxed{\mu = n}$ $\quad n = 1, 2, 3, \dots \qquad \lambda = -\mu^2 = -n^2 \quad n = 1, 2, 3, \dots$

For each n, get: $X_n(x) = \sin nx$

(the c_1 will return later).

Solve the T problem now. (Same ODE).

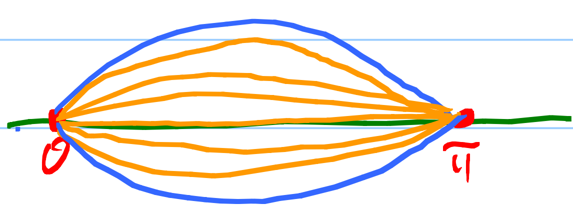
$$T_n(t) = c_1 \cos nt + c_2 \sin nt$$

Want $\frac{\partial y}{\partial t}(x,0) = 0$. Need $c_2 = 0$.

Get sol^n $\quad u_n(x,t) = X_n(x)\, T_n(t)$

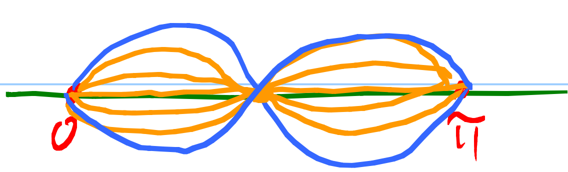
$$\boxed{u_n(x,t) = \sin nx \cos nt}$$

$n = 1$



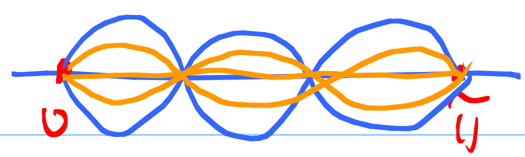
$\sin x \cos t$

$n = 2$

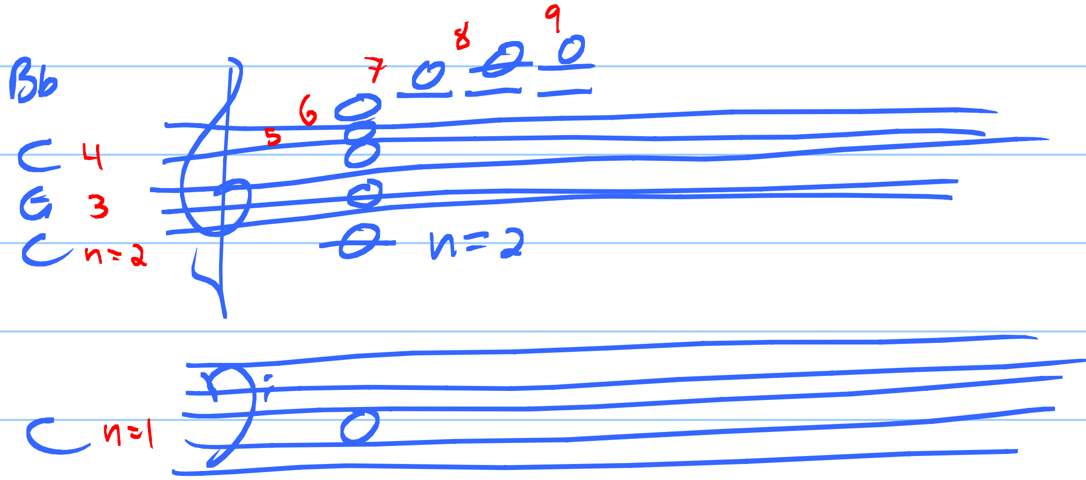


$\sin 2x \cos 2t$

$n=3$



$\sin 3x \ \cos 3t$



Big Idea: Linear Problem! Can add solutions!

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

Initial Cond: $u(x,0) \overset{\text{want}}{=} f(x)$

$$\sum_{n=1}^{\infty} c_n \sin nx = f(x)$$

Key Fact: $\sin nx$ are "orthogonal" functions.

$$(\sin nx, \sin mx) = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ \frac{\pi}{2} & \text{if } n = m \end{cases}$$

$$\left(\sum c_n \sin nx\right) \sin mx = f(x) \sin mx$$

Integrate $\displaystyle \sum_{n=0}^{\infty} c_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{\text{all zero except } n=m \text{ one!}} = \int_0^{\pi} f(x) \sin mx \, dx$

$$c_m \left(\frac{\pi}{2}\right) = \int_0^{\pi} f(x) \sin mx \, dx$$

$$\boxed{c_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$