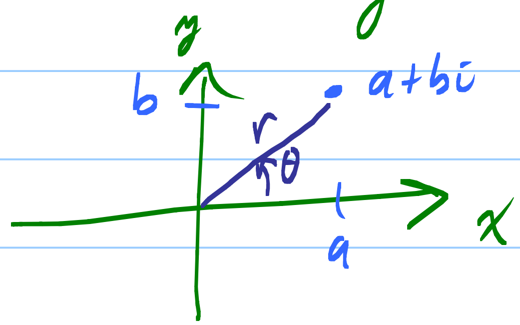
(Online course evaluations for C. He and me.)

Old Final Exam and Review Problems on Home Page.

Geometric meaning of complex multiplication.



$a = r\cos\theta$

$b = r\sin\theta$

$$a+bi = r\cos\theta + ir\sin\theta = r\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(-1)\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \cos\theta + i\sin\theta$$

Easy fact: Complex arithmetic satisfies all the rules of a field, for example

$z \cdot w = w \cdot z$ $\qquad$ $z\cdot(w+v) = z\cdot w + z \cdot v$

$(z\cdot w)\cdot v = z(w\cdot v)$ $\qquad$ etc.

Polar form: $a+bi = re^{i\theta}$

Big Fact: $(a+bi)^n = (re^{i\theta})^n = (re^{i\theta})(re^{i\theta})\cdots(re^{i\theta})$

$$= r^n \left(e^{i\theta}\right)^n$$

$$= r^n e^{in\theta}$$

Why: $e^{i\alpha} e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$= \underbrace{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}_{\cos(\alpha+\beta)} + i\underbrace{(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}_{\sin(\alpha+\beta)}$$

$$= e^{i(\alpha+\beta)}$$

So $e^{i\theta} e^{i\theta} = e^{i2\theta}$

$$\left(e^{i\theta} e^{i\theta}\right) e^{i\theta} = \left(e^{i2\theta}\right) e^{i\theta} = e^{i3\theta}$$

$$\vdots$$

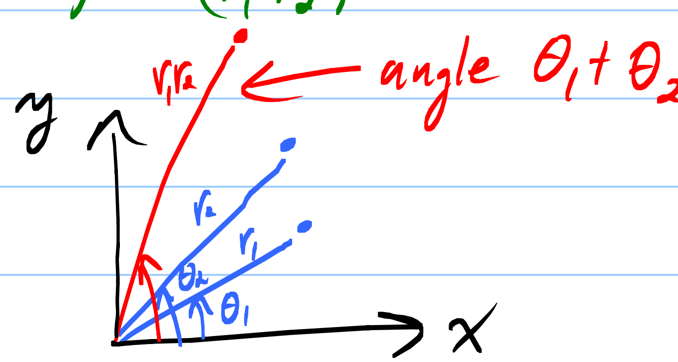
$$\left(e^{i\theta}\right)^n = e^{in\theta} \quad \checkmark$$

Geometric meaning of $z_1 \cdot z_2$:

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$

$$z_1 \cdot z_2 = \left(r_1 e^{i\theta_1}\right)\cdot\left(r_2 e^{i\theta_2}\right) = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$$

Picture:



Miracle of complex numbers: Tossing in just one extra number i that is a root of $x^2+1=0$, suddenly every polynomial has a full set of roots and can be factored over $\mathbb{C}$. (Fundamental Theorem of Algebra).

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n (z-r_1)(z-r_2)\cdots(z-r_N)$$

where r_k are (possibly repeated) numbers in $\mathbb{C}$.

Prob: Solve diff eqn $y^{(4)} + y = 0$.

Try $y = e^{rx}$. Plug in

$$(r^4 e^{rx}) + e^{rx} = 0 \quad \text{want}$$

$$\underbrace{(r^4+1)}_{\text{need } r^4+1=0} e^{rx} = 0$$

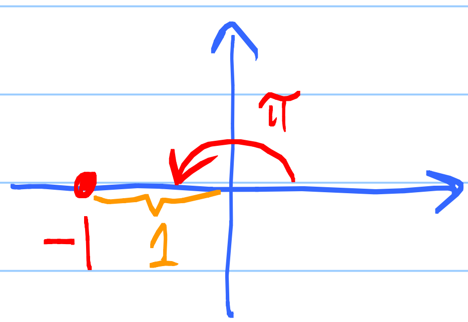
Prob: Find the four roots of $r^4+1=0$.

Let $r = Re^{i\theta}$. Want

$$r^4 = -1$$

$$(Re^{i\theta})^4 = -1$$

$$R^4 e^{i4\theta} = \underbrace{-1}_{1\cdot e^{i\pi}}$$



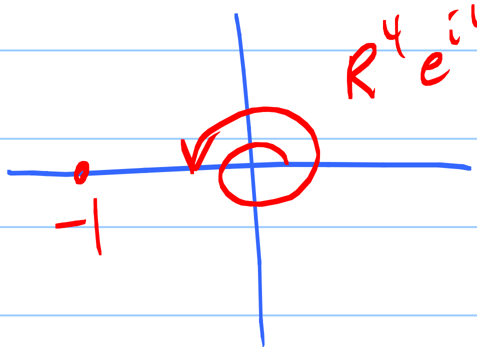
See R^4 = radius of $1 \cdot e^{i\pi}$. So $R^4 = 1$.

Since we are taking R real and >0, see $\boxed{R=1}$.

$4\theta = \pi$ gives a sol^n. $\theta = \frac{\pi}{4}$.

Root #1: $1 \cdot e^{i\pi/4}$. Check: $(e^{i\pi/4})^4 = e^{i\pi} = -1$ ✓

Need 3 more roots!



$R^4 e^{i4\theta}$ works too

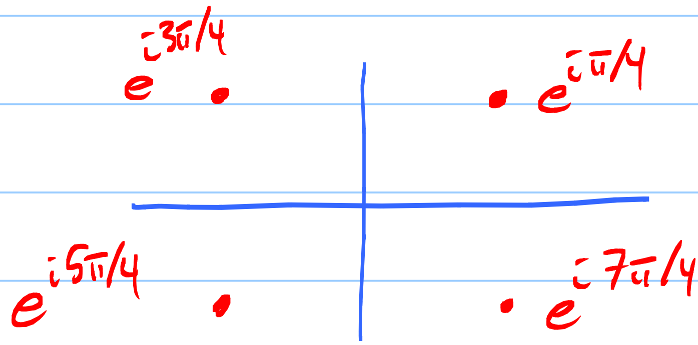
$4\theta = 2\pi + \pi = 3\pi$

$\theta = \frac{3\pi}{4}$

Root #2: $r = 1 \cdot e^{i3\pi/4}$

All the rest $4\theta = \pi + 2\pi n$ $n = 0, \pm 1, \pm 2, \ldots$

$\theta = \frac{\pi}{4} + \frac{\pi n}{2}$ $\underbrace{n = 0, \pm 1, \pm 2, \ldots}_{\text{ouch!}}$



∞ many? No! Just 4.

$$e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

4 roots: $\pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$.

$$r^4+1 = \left(r - \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right)\left(r - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\cdots\left(r - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)$$

Fact: If $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$, then

$r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi n$ for some

$n = 0, \pm 1, \pm 2, \ldots$

Sol^n to $y^{(4)} + y = 0$:

$$y = c_1 e^{\frac{\sqrt{2}}{2}x}\cos\frac{\sqrt{2}}{2}x + c_2 e^{\frac{\sqrt{2}}{2}x}\sin\frac{\sqrt{2}}{2}x$$
$$+ c_3 e^{-\frac{\sqrt{2}}{2}x}\cos\frac{\sqrt{2}}{2}x + c_4 e^{-\frac{\sqrt{2}}{2}x}\sin\frac{\sqrt{2}}{2}x$$

Read Appendix A-7.

Ap-34: 2b. $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1+i$

$\frac{1+i}{1-i} = (1+i)(1-i)^{-1}$

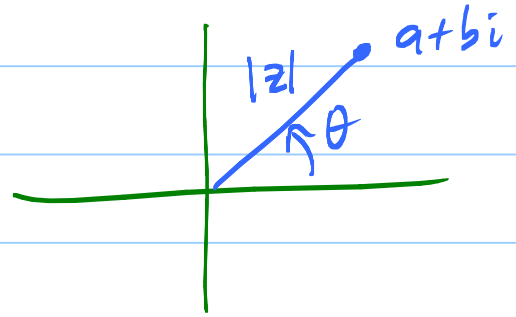
$\frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{2i}{1^2+1^2} = i$

$\frac{1}{x+iy} = 2+i$

$x+iy = \frac{1}{2+i}$

Note: $z = a+bi$, $\bar{z} = a-bi$, $|z| = \sqrt{a^2+b^2}$

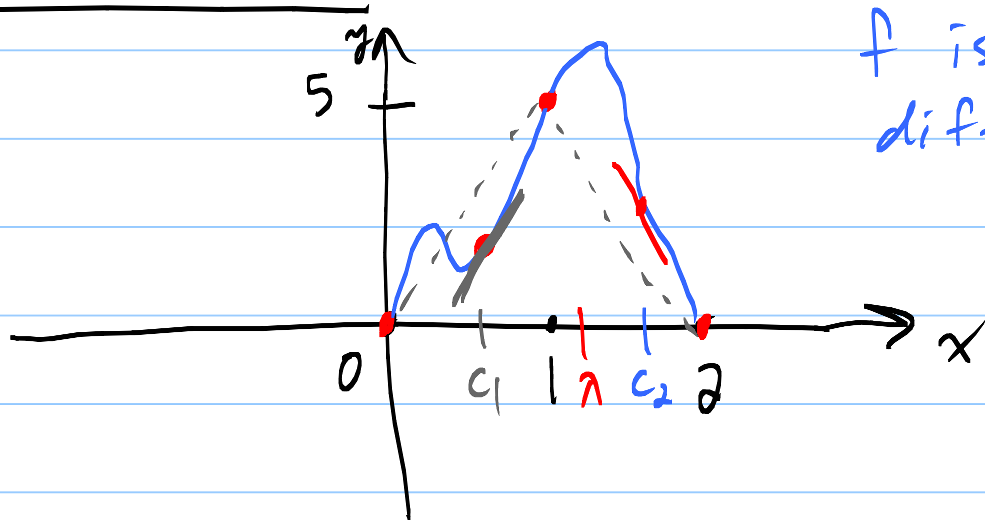
$$\frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{a-bi}{a^2+b^2}$$



$|z|$ = modulus of z

θ = argument of z

Prob 6 on old final:



f is twice diff'ble.

MVT: $\dfrac{f(1)-f(0)}{1-0} = \underbrace{\dfrac{5-0}{1-0}}_{5} = f'(c_1)$

$\dfrac{f(2)-f(1)}{2-1} = \underbrace{\dfrac{0-5}{2-1}}_{-5} = f'(c_2)$

Can't happen that $|f'(t)| < 5$ for all t.

$$\frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{-5-5}{c_2 - c_1} = f''(x)$$

$$f''(x) = \frac{-10}{c_2 - c_1} < \underbrace{\frac{-10}{2-0}}_{-5}$$

There must be a point where $f''(x) < -5$,

so $|f''(t)|$ can't be less than or equal to 5 for all t.