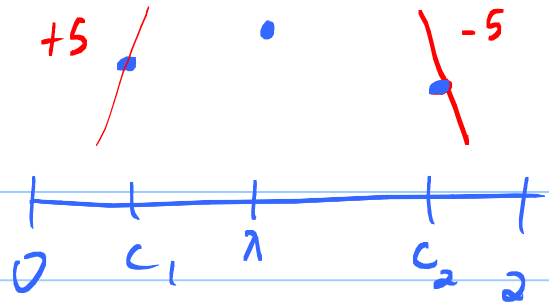
6.



$$f''(\lambda) = \frac{-5-5}{c_2 - c_1} \qquad 0 < c_2 - c_1 < 2$$

$$f''(\lambda) < -5$$

Precise statement: It can't happen that $|f'(x)| < 5$ for all x, and it can't happen that $|f''(x)| < 5$ for all x.

16. $\frac{1}{1-x} = ? \qquad |x| > 1$

Know $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$ when $|r| < 1$.

Trick: $\frac{1}{1-x} = \frac{1}{1-x} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{\frac{1}{x}}{\frac{1}{x} - 1}$

$= \frac{-\left(\frac{1}{x}\right)}{1 - \left(\frac{1}{x}\right)}$. Let $r = \frac{1}{x}$. $|x| > 1 \Rightarrow |r| < 1$

$= -r \cdot \left(\frac{1}{1-r}\right) = -r\left(1 + r + r^2 + \cdots\right)$

$= -r - r^2 - r^3 - \cdots$

$= -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots$

12. Taylor Series for $f(x) = \frac{1}{3-x}$ about $a = 1$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

One way: Compute $f^{(n)}(1)$ and do it.

2nd way: $\frac{1}{3-x} = \frac{1}{3-(x-1)-1} = \frac{1}{2-(x-1)} \cdot \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$

$= \frac{1/2}{1-\left(\frac{x-1}{2}\right)}$ Aha! Let $r = \left(\frac{x-1}{2}\right)$.

$$\frac{1}{3-x} = \frac{1}{2} \cdot \frac{1}{1-r} = \frac{1}{2}\left(1 + r + r^2 + r^3 + \cdots\right)$$

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{(x-1)}{2} + \frac{1}{2} \frac{(x-1)^2}{2^2} + \frac{1}{2} \frac{(x-1)^3}{2^3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (x-1)^n$$

4. $\frac{dy}{dx} = y(y+1)$ $\quad y(0) = 2$.

$$\frac{dy}{y(y+1)} = dx$$

$$\int \frac{1}{y(y+1)} dy = \int dx = x + C$$

Partial Frac: $\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$

Mult. by denom $y(y+1)$: $1 = A(y+1) + By$

$$0 \cdot y + 1 = \underbrace{(A+B)}_{0} y + \underbrace{(A)}_{1}$$

$A = 1$

$A+B=0 \Leftarrow B = -A = -1$. So $\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$.

$$\int \frac{1}{y} - \frac{1}{y+1}\, dy = x + C$$

$\ln y - \ln(y+1) = x + C \leftarrow$ Find C now!

Plug in $x=0$, $y=2$. $[y(0)=2]$

$\ln 2 - \ln 3 = 0 + C$ $\boxed{C = \ln \frac{2}{3}}$

$$\to \ln \frac{y}{y+1} = x + \ln \frac{2}{3}$$

$$\frac{y}{y+1} = e^{x + \ln \frac{2}{3}} = e^{x} e^{\ln \frac{2}{3}} = \frac{2}{3} e^{x}$$

$$y = \left(\tfrac{2}{3}e^{x}\right) y + \tfrac{2}{3} e^{x}$$

$$y\left(1-\frac{2}{3}e^x\right) = \frac{2}{3}e^x$$

$$\boxed{y = \frac{\frac{2}{3}e^x}{1-\frac{2}{3}e^x}}$$

2a) $I = \int \frac{x^3}{(1+x^2)^{1/3}}\,dx$ Hint: Let $u = 1+x^2$

$du = 2x\,dx$

$\boxed{x\,dx = \frac{1}{2}du}$

$$= \int \frac{\overbrace{x^2}^{=u-1}}{u^{1/3}} \underbrace{(x\,dx)}_{\frac{1}{2}du}$$

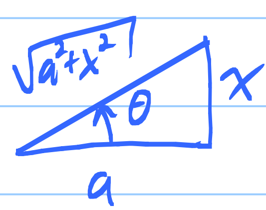
$$= \int \frac{u-1}{u^{1/3}}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{2/3} - u^{-1/3}\,du$$

$$= \frac{1}{2}\left(\frac{1}{\frac{2}{3}+1}u^{\frac{2}{3}+1} - \frac{1}{-\frac{1}{3}+1}u^{-1/3+1}\right) + C$$

$u = 1+x^2$

done!

2b) $I = \int \frac{1}{(1+x^2)^2}\,dx$ a^2+x^2 $x = a\tan\theta$

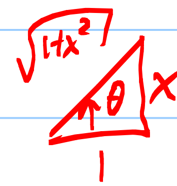


$$\frac{\sqrt{a^2+x^2}}{a} = \sec\theta \qquad dx = a\sec^2\theta\, d\theta$$

$$I = \int \frac{1}{(\sec^2\theta)^2}\left(\sec^2\theta\, d\theta\right) = \int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \int \cos^2\theta\, d\theta = \int \tfrac{1}{2}\left(1+\cos 2\theta\right) d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\underbrace{\sin 2\theta} + C$$



$\theta = \operatorname{Tan}^{-1} x$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\cdot\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

3. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

Ratio Test : $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{\left(\frac{x^{n+1}}{n+1}\right)}{\left(\frac{x^n}{n}\right)}\right| = \dfrac{n}{n+1}|x| \to |x|$

as $n\to\infty$.

$|x| < 1$ absolute convergence

$|x| > 1$ divergence

Test fails at $x = \pm 1$.

$\underline{x=1}$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges! p-series, $p=1$.

$\underline{x=-1}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alt. series test.

Since $\sum \frac{1}{n}$ diverges, the convergence is conditional at $x=-1$.

b) $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \leftarrow f(0)=0$

$$f'(x) = 1 + \frac{2x}{2} + \frac{3x^2}{3} + \frac{4x^3}{4} + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{1-x}$$

So $f(x) = \int \frac{1}{1-x}\,dx = -\ln|1-x| + C$

$f(0)=0$ shows $C=0$.

OR $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

$$= \int 1 + x + x^2 + x^3 + \cdots \, dx$$

$$= \int \frac{1}{1-x}\,dx = -\ln|1-x| + C$$

Get $C=0$ as above.