Solutions to Exams 1 & 2

MyLab HW 36 & GS HW 34, 35, 36

Final exam: Wed, May 6, 8:00-10:00 am in Elliott Hall of Music,
20 question multiple choice exam covering whole course,
find Practice problems at www.math.purdue.edu/MA266
Prac prob 1-16 (Exm 1), 17-29 (Exm 2), 30-35 Laplace transf.
Find answers on last page

1. Find the general solution to the homogeneous differential equation

$\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)$

$v^2 + v$

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad x > 0,$$

homogeneous!

by using substitution $v = y/x$.

Ans: $y = \dfrac{x}{C - \ln x}$

$v = \frac{y}{x}$ So $y = xv$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

Get new eqn: $x\frac{dv}{dx} + v = v^2 + v$

$$x\frac{dv}{dx} = v^2$$

separable

$$\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{v} = \ln x + C$$

$$\frac{1}{\left(\frac{y}{x}\right)} = -\ln x - C$$

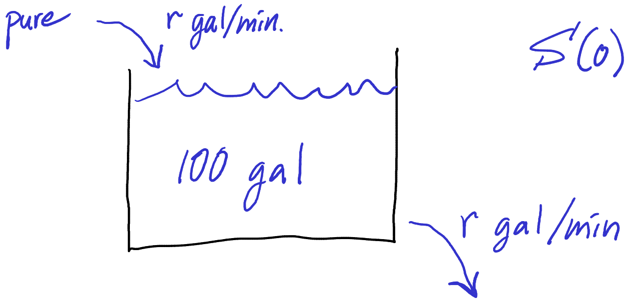
$$y = \frac{x}{-\ln x - C}$$

Let $\tilde{C} = -C$ to get ans above

✓

2. A full 100-gallon tank contains 50 pounds of dissolved salt. Pure water is pumped into the well-stirred tank at a rate of r gallons per minute, while the mixture is simultaneously drained at the same rate. After 10 minutes, only 30 pounds of salt remain. Find r.

Ans: $r = 10\ln(5/3)$



$S(0) = \underline{50 \text{ lbs}}$

$$\frac{dS}{dt} = (\text{RATE IN}) - (\text{RATE OUT})$$

$$\frac{dS}{dt} = r \cdot \overset{\text{pure}}{0} - r \cdot \frac{S(t)}{100}$$

depends on t!

$100 \leftarrow V(t)$

1. $\frac{dS}{dt} + \frac{r}{100} S = 0$ Int factor: $u = e^{\int \frac{r}{100} dt} = e^{\frac{r}{100}t}$

$P(t) = \frac{r}{100}$

$$\underbrace{e^{\frac{r}{100}t}\left(\frac{dS}{dt} + \frac{r}{100} S\right)}_{\frac{d}{dt}\left[e^{\frac{r}{100}t} S\right]} = 0 \cdot e^{\frac{r}{100}t} = 0$$

So $e^{\frac{r}{100}t} S = \int 0 \, dt = C$

$$S(t) = Ce^{-\frac{r}{100}t}$$

want

$$S(0) = Ce^0 = 50$$ So $\underline{C = 50}$

So $S(t) = 50 e^{-\frac{r}{100}t}$

Know $S(10) = 50 e^{-\frac{10}{100}r} = 30$

$$e^{-r/10} = \frac{3}{5}$$

$$-\frac{r}{10} = \ln \frac{3}{5}$$

$$r = -10 \ln \frac{3}{5} = 10 \ln \left(\frac{3}{5}\right)^{-1}$$

$$= 10 \ln \frac{5}{3}$$ ✓

3. Solve the following initial value problem

Ans: $y = \frac{e^x}{2-e^x}$

$$\frac{dy}{dx} = y^2 + y, \quad y(0) = 1.$$

<u>Method 1</u>

Separable. $\int \frac{1}{y(y+1)}\,dy = \int dx$

$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$

$1 = A(y+1) + By = \underbrace{(A+B)}_{0}y + \underset{1}{\overset{\shortparallel}{A}}$

$B = -A = -1$

$$\int \frac{1}{y} - \frac{1}{y+1}\,dy = x + C$$

$$\ln y - \ln(y+1) = x + C$$

$y=1$ when $x=0$: $\underbrace{\ln 1}_{0} - \ln 2 = 0 + C.$ $\boxed{C = -\ln 2}$

$$\ln \frac{y}{y+1} = x - \ln 2$$

$$\frac{y}{y+1} = e^{x-\ln 2} = e^x e^{-\ln 2} = e^x e^{\ln 2^{-1}} = \frac{1}{2}e^x$$

$$y = (y+1)\frac{1}{2}e^x$$

$$y\left(1 - \frac{1}{2}e^x\right) = \frac{1}{2}e^x$$

$$y = \frac{\frac{1}{2}e^x}{1 - \frac{1}{2}e^x} = \frac{e^x}{2-e^x}$$

<u>Method 2</u> Bernoulli eqn $1 \cdot \frac{dy}{dx} - y = y^2$ ← $n = 2$

$P(x) = -1$

$Q(x) = 1$

In <u>standard form</u> $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$

Change of variables: $v = y^{1-n}$ yields <u>new</u> ODE

$$\boxed{\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)\,Q(x)}$$

So $v = y^{1-2} = \frac{1}{y}$. $\frac{dv}{dx} + (1-2)(-1)\,v = (1-2)\cdot 1$

$$\frac{dv}{dx} + v = -1 \qquad \text{Linear: Int factor} = e^x$$

$$e^x\left(\frac{dv}{dx}+v\right) = -1\cdot e^x$$

$$[e^x v]' = -e^x$$

$$e^x v = \int -e^x\,dx = -e^x + C$$

$$v = \frac{C-e^{-x}}{e^x}$$

$$v = \frac{1}{y} = \frac{C-e^{-x}}{e^x}$$

$y=1$ when $x=0$: $\quad \frac{1}{1} = \frac{C-1}{1} \qquad C=2$

$$y = \frac{e^x}{2-e^x} \quad \checkmark$$

4. Find the value of the parameter α for which the equation is exact. For the value of α for which the equation is exact, find an implicit solution of the initial value problem:

$$\underbrace{(2xe^{2y} - \sin(x))}_{M}dx + \underbrace{(\alpha x^2 e^{2y} + 3y^2)}_{N}dy = 0, \quad y(0) = 1.$$

Ans: $\alpha = 2$, $x^2e^{2y} + \cos x + y^3 = 2$

Need $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \leftarrow \frac{\partial}{\partial y}(dx\text{ thing}) = \frac{\partial}{\partial x}(dy\text{ thing})$

$$4xe^{2y} - 0 = 2\alpha x e^{2y} + 0 \qquad \boxed{\alpha = 2} \checkmark$$

A) $\frac{\partial \varphi}{\partial x} = M = 2xe^{2y} - \sin x \leftarrow$ So $\varphi = \int 2xe^{2y} - \sin x\, dx$

$= x^2e^{2y} + \cos x + C(y)$

unknown fcn of y

B) $\frac{\partial \varphi}{\partial y} = N = 2x^2e^{2y} + 3y^2$

Use (B) in a different way:

$$\frac{\partial}{\partial y}\Big[\underset{\substack{\uparrow\\ \text{from (A)}}}{x^2e^{2y}+\cos x}+C(y)\Big] \overset{\text{want}}{=} 2x^2e^{2y}+3y^2$$

$$x^2 2e^{2y}+0+C'(y) = 2x^2e^{2y}+3y^2$$

$$\boxed{C'(y)=3y^2}$$

$$C(y)=y^3 \leftarrow \text{no } +K \text{ here.}$$

So $\varphi = x^2e^{2y}+\cos x+y^3 = K \leftarrow$ important!

$y=1$ when $x=0$: $\quad 0^2\cdot e^{2\cdot 1}+\cos 0+1^3 = K$

$$0 + 1 + 1 = K \qquad K=2$$

Ans: $x^2e^{2y}+\cos x+y^3 = 2$ ✓

5. Consider the first-order linear initial value problem

$$\frac{dy}{dx} = \overbrace{2xy-4x}^{f(x,y)}, \quad y(\underset{x_0=0}{0}) = \underset{y_0=1}{1}.$$

Ans: -13

Apply Euler's method with step size $h=1$ to approximate $y(3)$.

$\frac{dy}{dx}=f(x,y)$ $\qquad \boxed{y_{n+1}=y_n+hf(x_n,y_n)}$ $\qquad x_{n+1}=x_n+h$

$x_0=0$, $y_0=1$ and $f(x,y)=2xy-4x$

n	x_n	y_n	$y_{n+1}=y_n+2x_ny_n-4x_n$
0	0	1	$1 = 1+2\cdot 0\cdot 1-4\cdot 0=1$
1	1	1	$-1 = 1+2\cdot 1\cdot 1-4\cdot 1=-1$
2	2	-1	$-13 = (-1)+2\cdot 2\cdot(-1)-4\cdot 2=-13$
3	3	-13	

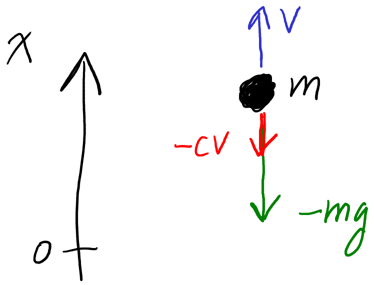
$y(3)\approx -13$ ✓

Improved Euler method: $\underline{\text{Step 1}}$: Euler $\tilde{y}_{n+1} = y_n + h f(x_n, y_n)$

$\underline{\text{Step 2}}$ Improved Euler $y_{n+1} = y_n + h\left[\dfrac{f(x_n,y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2}\right]$

6. A particle of mass m moves vertically under the influence of gravity and air resistance. Air resistance is twice the magnitude of the velocity and acts in the opposite direction of motion. Let $v(t)$ be the velocity of the particle, where upward direction is positive. Which differential equation correctly models the particle's velocity? Here $g > 0$ is the acceleration due to gravity.

Ans: $m\dfrac{dv}{dt} = -mg - 2v$



$ma = F_{net}$

$m\dfrac{dv}{dt} = -mg - cv$ ✓

7. Find a form for the general solutions to the differential equation

$$y^{(4)} - 6y^{(3)} + 9y'' = 0.$$

with independent variable x. Ans: $c_1 + c_2 x + c_3 e^{3x} + c_4 x e^{3x}$

$r^4 - 6r^3 + 9r^2 = 0$

$r^2(r^2 - 6r + 9) = 0$

$r^2(r-3)^2 = 0$ $\quad r = 0, 0, 3, 3$

4 sol'ns $\left\{ \underbrace{e^{0\cdot x}}_{1},\ e^{3x},\ xe^{0\cdot x},\ xe^{3x} \right.$

Gen'l sol'n $y = c_1 \underbrace{e^{0\cdot x}}_{1} + c_2 x \underbrace{e^{0\cdot x}}_{1} + c_3 e^{3x} + c_4 x e^{3x}$ ✓

8. For $x > 0$, solve the initial value problem

$$xy' + (1+x)y = x, \quad y(1) = 1.$$

Linear. 1) Put in standard form

$$1 \cdot \frac{dy}{dx} + \underbrace{\left(1 + \frac{1}{x}\right)}_{P(x)} y = \underbrace{1}_{Q(x)}$$

2) Int factor: $u = e^{\int P(x)\,dx} = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x}$

$$= e^x e^{\ln x} = x e^x$$

3) Multiply standard form eqn by u:

$$\underbrace{x e^x \left(y' + (1+\tfrac{1}{x}) y \right)} = 1 \cdot x e^x$$

$$\left[x e^x \cdot y \right]' = x e^x$$

4) Integrate:

$$x e^x y = \int \underset{u}{x}\, \underbrace{e^x\,dx}_{dv}$$

$\boxed{\begin{array}{l} u = x \\ du = dx \\ v = \int e^x dx = e^x \end{array}}$

$$= uv - \int v\,du$$

$$= x e^x - \int e^x dx$$

$$x e^x y = x e^x - e^x + C$$

5) Solve for y ... or get C now. $y(1) = 1$

$y = 1$ when $x = 1$

$$1 \cdot e^1 (1) = 1 \cdot e^1 - e^1 + C$$

$$e = e - e + C \qquad \boxed{C = e}$$

6) Now solve for y: $y = \dfrac{x e^x - e^x + e}{x e^x} = 1 - \dfrac{1}{x} + \dfrac{e}{x} e^{-x}$

9. Consider a population $x(t)$ satisfying the differential equation

$$\frac{dx}{dt} = x(6-x) - 8.$$

(a) Find all critical points of the equation and draw the phase diagram. Classify the stability of each equilibrium solution. Ans: $x=2$ (stable), $x=4$ (unstable)

(b) Sketch a few solution curves, including the equilibrium solutions you found in part (a).

(c) If the initial population is $x(0)=3$, describe the asymptotic behavior of the solution as $t \to \infty$.

$\lim_{t\to\infty} x(t) = 4$

($x(t) \nearrow 4$ but never gets there.)

Critical points: $x(6-x) - 8 = 0$

$$-x^2 + 6x - 8 = 0$$

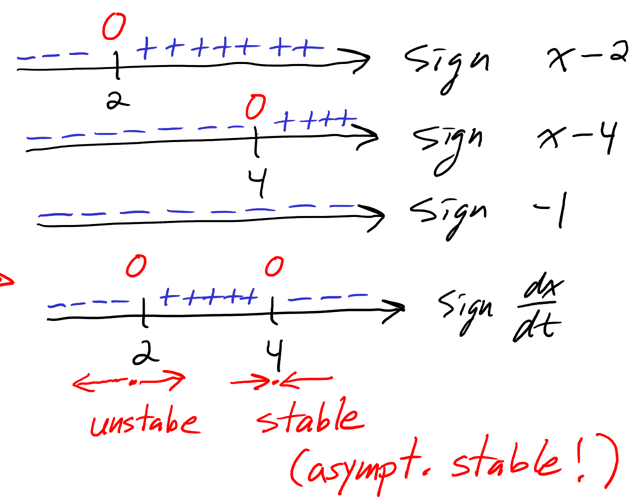
$$-\left[x^2 - 6x + 8 = 0\right]$$

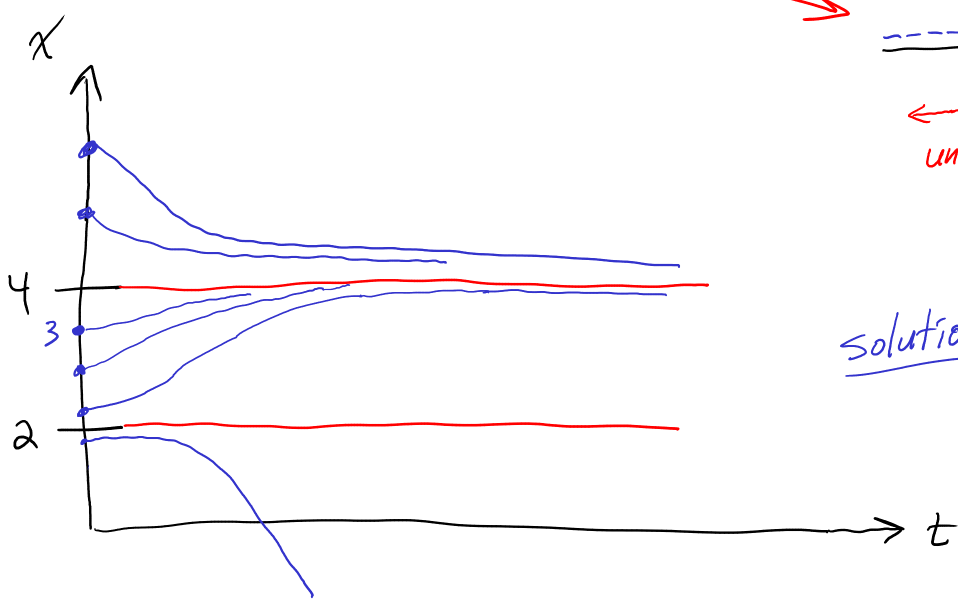
$$-\left[(x-2)(x-4) = 0\right]$$

Crt. pts. $x = 2, 4$

Sign of $\frac{dx}{dt} = -(x-2)(x-4)$

Phase diagram:





solutions neve cross

10. Solve the initial value problem

$$y'' - 2y' + 17y = 0, \quad y(0) = -2,\ y'(0) = 6.$$

$$r^2 - 2r + 17 = 0$$

$$r = \frac{2 \pm \sqrt{4-68}}{2} = 1 \pm \frac{\sqrt{-64}}{2} = 1 \pm \frac{8i}{2}$$

$$= 1 \pm 4i$$

Complex sol^n: $e^{(1+4i)x} = e^x e^{4ix}$

$= e^x(\cos 4x + i\sin 4x)$

$= \underbrace{e^x \cos 4x}_{y_1} + i\,\underbrace{e^x \sin 4x}_{y_2}$

Two real sol^{n}s

Gen^l sol^n

$y = c_1 e^x \cos 4x + c_2 e^x \sin 4x$

$y(0) = c_1 \overset{\text{want}}{=} -2$

$y' = c_1(e^x\cos 4x - 4e^x \sin 4x) + c_2(e^x \sin 4x + 4e^x \cos 4x)$

$y'(0) = c_1 + 4c_2 \overset{\text{want}}{=} 6$

$\boxed{c_1 = -2}$

$4c_2 = 6 - (-2) = 8$

$\boxed{c_2 = 2}$

Ans: $y = -2e^x \cos 4x + 2e^x \sin 4x$

Exam 2

1. A mass of 3 kg is attached to a spring with spring constant 5 N/m and a dashpot with damping constant $c > 0$. For which values of c will the system be overdamped?

Ans: $c > 2\sqrt{15}$

$ma = -ky - cv$

$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$

$m = 3$ kg

$k = 5$ Nt/m

c in Nt/(m/sec) > 0

Roots: $r = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m} = \dfrac{-c \pm \sqrt{c^2 - 4\cdot 3\cdot 5}}{2\cdot 3}$

overdamped: r_1, r_2 real and unequal (both negative)

So need $c^2 > 4\cdot 3\cdot 5$

$c > \sqrt{4\cdot 3\cdot 5}$, i.e. $c > 2\sqrt{15}$

2. The correct *form* of a particular solution to

$$y'' + 4y' + 4y = x^2 e^{-2x}$$

(poly deg 2) $\cdot e^{-2x}$

Table : $(Ax^2+Bx+C)e^{-2x}$

gen^l poly deg 2

to be used in the method of undetermined coefficients is

Ans: $Ax^4e^{-2x} + Bx^3e^{-2x} + Cx^2e^{-2x}$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \qquad r = -2, -2$$

Homog sol^n: $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

y_p from table in Method of Undetermined Coeff:

$$y_p = (\text{General poly of degree } 2)\, e^{-2x}$$

$$= Ax^2e^{-2x} + Bxe^{-2x} + Ce^{-2x}$$

Oops. These solve homog.

Try $x \cdot$ (sol^n from table)

$$= Ax^3e^{-2x} + Bx^2e^{-2x} + Cxe^{-2x}$$

Oops. Solves homog

Finally, x^2 (sol^n from table) is safe

$$y_p = Ax^4e^{-2x} + Bx^3e^{-2x} + Cx^2e^{-2x} \quad \checkmark$$

3. Consider the mass-spring system with position $x(t)$ that satisfies the equation

$$x'' + 100x = 12\cos(\omega t), \quad x(0) = 0,\ x'(0) = 0.$$

Which of the following graphs best illustrates the behavior of the solution $x(t)$ for the value $\omega = 10$? resonance!

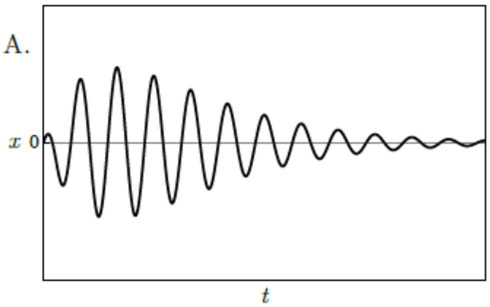
$r^2 + 100 = 0$

$r = \pm 10i$

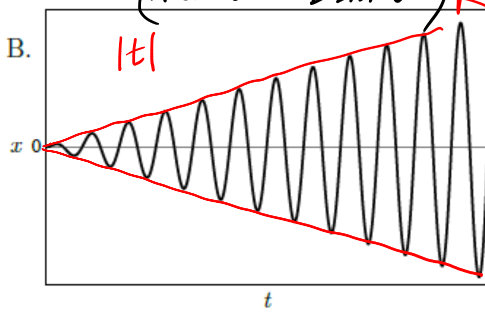
$\omega = 10$ = "natural frequency"

$y = c_1 \cos 10t + c_2 \sin 10t + (A\cos\omega t + B\sin\omega t)$ $\omega \neq 10$

○ A.



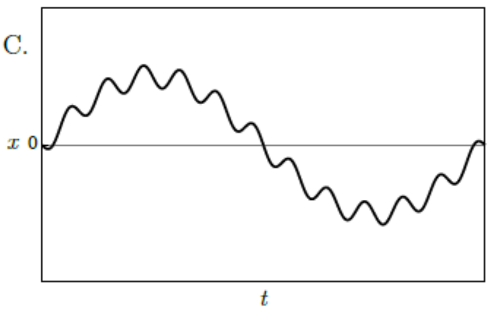
● B.



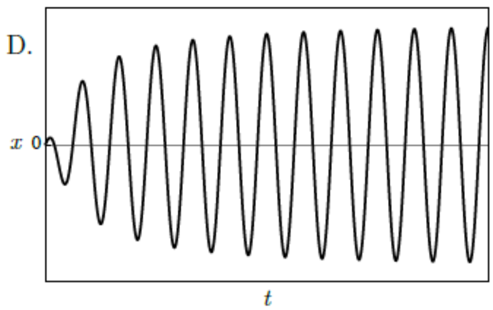
$|t|$

mult by t if $\omega = 10$

○ C.

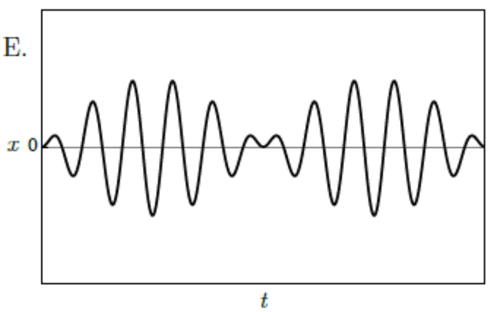


○ D.



friction

○ E.



$\omega \neq 10$

4. Transform the differential equation $x'' + 3x' + 2x = \cos(t)$ into an equivalent system of first order differential equations in x_1 and x_2 where $x_1 = x$ and $x_2 = x'$. Then write this system in matrix notation as $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$. The correct $\mathbf{A}$ and $\mathbf{f}(t)$ are:

$$x_1 = x(t)$$
$$x_2 = \dot{x}(t)$$

$$\begin{cases} x_1' = \dot{x} = x_2 \leftarrow \text{easy. Always same} \\ x_2' = \ddot{x} \underset{\uparrow \text{from ODE}}{=} -3\dot{x} - 2x + \cos t \\ \quad = -3x_2 - 2x_1 + \cos t \end{cases}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 3x_2 + \cos t \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

5. For the linear system

$$\mathbf{x}' = \begin{bmatrix} 4 & k \\ 2 & 0 \end{bmatrix} \mathbf{x},$$

<u>Ans</u>: $k < -2$

find all values for the parameter k such that the origin is a spiral source.

$$\det \begin{bmatrix} 4-r & k \\ 2 & 0-r \end{bmatrix} = r^2 - 4r - 2k = 0$$

$$r = \frac{4 \pm \sqrt{16+8k}}{2}$$

For a spiral, roots r need to be <u>complex</u>, so

$$16 + 8k < 0$$

$$k < \frac{-16}{8}$$

$$\boxed{k < -2}$$

$e^{2t}\cos bt$ and $e^{2t}\sin bt$

Roots : $2 \pm bi$ $\quad b = \sqrt{|16+8k|} \leftarrow k < -2$

real part +, so spiral is <u>out</u>. Source ✓

6. Find a general solution of the following system of first order differential equations

$$\mathbf{x}' = \begin{bmatrix} 7 & 1 \\ -12 & -1 \end{bmatrix} \mathbf{x}. \qquad \text{Ans: } c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -6 \end{pmatrix} e^{t}$$

$$\det \begin{bmatrix} 7-r & 1 \\ -12 & -1-r \end{bmatrix} = (r^2 - 6r - 7) - (-12) = r^2 - 6r + 5$$

$$= (r-5)(r-1) \qquad r = 1, 5 \leftarrow \text{eigenvalues}$$

For $r = 5$

$$(A - \underset{\uparrow r=5}{r} I)\vec{a} = \vec{0} \longrightarrow \left[\begin{array}{cc|c} 7-5 & 1 & 0 \\ -12 & -1-5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ & & \end{array}\right]$$

Eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

↑ all other e-vects are c times this, $c \neq 0$.

For $r = 1$

$$\left[\begin{array}{cc|c} 7-1 & 1 & 0 \\ -12 & -1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 6 & 1 & 0 \\ & & \end{array}\right]$$

Eigenvector: $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$

$$\text{Ans: } c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -6 \end{pmatrix} e^{t}$$

7. Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

$$\det \begin{bmatrix} -1-r & -4 \\ 1 & -1-r \end{bmatrix} = r^2 + 2r + 1 - (-4) = r^2 + 2r + 5$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

For $r = -1 + 2i$

Take one complex root.

$$\left[\begin{array}{cc|c} -1-(-1+2i) & -4 & 0 \\ \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2i & -4 & 0 \\ \end{array}\right]$$

Complex eigenvector $\begin{pmatrix} 4 \\ -2i \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -i \end{pmatrix}$ or $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$ or ...

Complex sol^n $\left[\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i\right] \left(e^{-t}\cos 2t + i e^{-t} \sin 2t\right) \leftarrow e^{rt}$

$$= \Bigg(\underbrace{\begin{pmatrix} 4 \\ 0 \end{pmatrix} e^{-t}\cos 2t \overset{i^2=-1}{-} \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{-t}\sin 2t} \Bigg) + \underline{\underline{i}} \Bigg(\underbrace{\begin{pmatrix} 4 \\ 0 \end{pmatrix} e^{-t}\sin 2t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{-t}\cos 2t} \Bigg)$$

two real sol^{n}s

$$\vec{x} = c_1 \begin{pmatrix} 4\cos 2t \\ 2\sin 2t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4\sin 2t \\ -2\cos 2t \end{pmatrix} e^{-t}$$

$$\vec{x}(0) = c_1\begin{pmatrix}4\\0\end{pmatrix} + c_2\begin{pmatrix}0\\-2\end{pmatrix} \overset{\text{want}}{=} \begin{pmatrix}2\\2\end{pmatrix}$$

$$c_1 = 1/2 \qquad c_2 = -1$$

Ans

$$\vec{x} = \frac{1}{2}\begin{pmatrix}4\cos 2t\\2\sin 2t\end{pmatrix}e^{-t} - \begin{pmatrix}4\sin 2t\\-2\cos 2t\end{pmatrix}e^{-t}$$

$$= \begin{pmatrix}2\cos 2t - 4\sin 2t\\ \sin 2t + 2\cos 2t\end{pmatrix}e^{-t}$$

8. Consider the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}$. Two linearly independent solutions for this system are given:

Gen'l sol^n: $\vec{x} = \mathbb{X}(t)\begin{pmatrix}c_1\\c_2\end{pmatrix}$

$\vec{x}(0) = \mathbb{X}(0)\vec{c} = \vec{x}_0$

$$\mathbf{x}_1(t) = \begin{bmatrix}e^{-2t}\\-3e^{-2t}\end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix}2e^{5t}\\e^{5t}\end{bmatrix}.$$

Find the matrix exponential $e^{\mathbf{A}t}$.

Fundamental matrix: $\mathbb{X}(t) = [\vec{x}_1, \vec{x}_2] = \begin{bmatrix}e^{-2t} & 2e^{5t}\\-3e^{-2t} & e^{5t}\end{bmatrix}$

same thing

Normalized Fundamental matrix: $\Phi(t) = \mathbb{X}(t)\mathbb{X}(0)^{-1}$

$\Phi(0) = I$

Matrix exponential $e^{A(t)} = \Phi(t)$

Note: $\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$, so

$\mathbb{X}(0)^{-1} = \begin{bmatrix}1 & 2\\-3 & 1\end{bmatrix}^{-1} = \underbrace{\frac{1}{1-(-6)}}_{1/7}\begin{bmatrix}1 & -2\\3 & 1\end{bmatrix}$ and

$$e^{\mathbb{A}t} = \Phi(t) = \mathbb{X}(t)\mathbb{X}(0)^{-1} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1/7 & -\frac{2}{7} \\ 3/7 & 1/7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7}e^{-2t} + \frac{6}{7}e^{5t} & -\frac{2}{7}e^{-2t} + \frac{2}{7}e^{5t} \\ \underline{-\frac{3}{7}e^{-2t} + \frac{3}{7}e^{5t}} & \underline{\frac{6}{7}e^{-2t} + \frac{1}{7}e^{5t}} \end{bmatrix} \quad \checkmark$$

(row 2)·(col 1) (row 2)·(col 2)

<u>Note</u> $e^{\mathbb{A}t}$ is also $= \mathbb{I} + \frac{1}{1!}\mathbb{A}t + \frac{1}{2!}\mathbb{A}^2t^2 + \cdots$

but this not the way to compute it. However, in cases

like $\mathbb{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = 2\mathbb{I} + \mathbb{N}$

diagonal matrix $\mathbb{D}$

nilpotent matrix $\mathbb{N}$

$\mathbb{N}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

we can use

1) $e^{(\mathbb{A}+\mathbb{B})t} = e^{\mathbb{A}t} \cdot e^{\mathbb{B}t}$ if $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$ <u>commute</u>.

2) Diagonal matrices commute with <u>any</u> matrix

3) So $e^{\mathbb{A}t} = e^{(\mathbb{D}+\mathbb{N})t} = e^{\mathbb{D}t} e^{\mathbb{N}t}$

$$= \begin{bmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{bmatrix} \cdot \left(\mathbb{I} + \mathbb{N}t + \frac{1}{2!}\underbrace{\mathbb{N}^2}_{\text{0 matrix}} t^2 + \cdots \right)$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} t + \text{zero matrices} \right)$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ -te^{2t} & e^{2t} \end{bmatrix}$$

9. Use the *method of variation of parameters* to find a general solution to the differential equation

$$1.\ y'' - 2y' + y = \frac{e^x}{x^2}, \quad x > 0.$$

standard form ✓ $F(x)$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1, 1$

Homog sol^n: $y = c_1 \underbrace{e^x}_{y_1} + c_2 \underbrace{xe^x}_{y_2}$

$$W[y_1, y_2] = W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} = e^{2x}$$

(row 2) − (row 1) = $[0, e^x]$

Particular sol^n: $y_p = u_1 y_1 + u_2 y_2$

where
$$\begin{cases} u_1' = \dfrac{-y_2 F}{W} = \dfrac{(-xe^x)\left(\frac{e^x}{x^2}\right)}{e^{2x}} = -\dfrac{1}{x} & u_1 = \int -\frac{1}{x}\,dx = -\ln x \\[2ex] u_2' = \dfrac{y_1 F}{W} = \dfrac{(e^x)\left(\frac{e^x}{x^2}\right)}{e^{2x}} = \dfrac{1}{x^2} & u_2 = \int \frac{1}{x^2}\,dx = -\frac{1}{x} \end{cases}$$

Gen^l sol^n

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + (u_1 y_1 + u_2 y_2)$$

$$= c_1 e^x + c_2 x e^x + (-\ln x) e^x + \underbrace{\left(-\frac{1}{x}\right)}_{u_2} \underbrace{x e^x}_{y_2} \quad ✓$$

$$= \underbrace{(c_1 - 1)}_{\tilde{c}_1} e^x + c_2 x e^x \underbrace{- e^x \ln x}$$

easiest $y_p = -e^x \ln x$

10. Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\det \begin{bmatrix} -1-r & 1 \\ -1 & -3-r \end{bmatrix} = r^2 + 4r + 3 - (-1) = r^2 + 4r + 4$$

$$= (r+2)^2 = 0 \qquad r = -2, -2$$

For $r = -2$

$$\left[\begin{array}{cc|c} -1-(-2) & 1 & 0 \\ & & \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ & & \end{array}\right]$$

Eigenvector $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Get $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$

Only one linearly independent e-vect for $r = -2$.

So second sol^n is $\vec{x}_2 = \vec{a}\, t e^{-2t} + \vec{b} e^{-2t}$ where

$$\underbrace{(A - rI)}_{r=-2} \vec{b} = \vec{a} \qquad \left[\begin{array}{cc|c} -1-(-2) & 1 & 1 \\ -1 & -3-(-2) & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right] \leftarrow \text{lots of sol}^n\text{s.}$$

Pick nice one $b_1 = 1$, $b_2 = 0$.

$$\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}$$

Gen' Sol^n $\quad \vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \right]$

want

Want $\quad \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$c_1 = -1 \qquad c_2 = 2 - c_1 = 2 - (-1) = 3$

$$\vec{x}(t) = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + 3 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \right]$$

$$= \begin{pmatrix} 3t + 2 \\ -3t + 1 \end{pmatrix} e^{-2t}$$