Lecture 2 MA 266-111 Steve Bell

Section 1.2 on assignment sheet

Last time: Easiest ODE: $\frac{dy}{dx} = 2x$

$y = \int 2x\,dx = x^2 + C$ $\quad C = y(0)$

This time: $\frac{dy}{dx} = ky \leftarrow$ y is an unknown fcn of x!

$\underbrace{\frac{1}{y}\frac{dy}{dx}}_{\frac{d}{dx}\left(\ln|y|\right)} = k$

$\int \frac{1}{u}\,du = \ln|u| + C$

$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}\ln|u|$

Sol^n: $e^{\ln|y|} = e^{\int k\,dx} = e^{kx + C}$

$|y| = e^{kx+C} = e^{kx}e^{C}$

$y = \underbrace{\left(\pm\, e^{C}\right)}_{\tilde{C}} e^{kx}$

$\tilde{C} \leftarrow$ anything except 0

Tradition: Call $\tilde{C}$ C now

Sol^n: $y = Ce^{kx}$

Solution means: $\frac{dy}{dx} = ky$

$$\frac{d}{dx}\Big(\underset{Ce^{kx}}{\underset{\uparrow}{eeee}}\Big) = k\Big(\underset{Ce^{kx}}{\underset{\uparrow}{eeee}}\Big)$$

$$Cke^{kx} = kCe^{kx} \quad \checkmark$$

Hmmm. Works even if $C=0$!

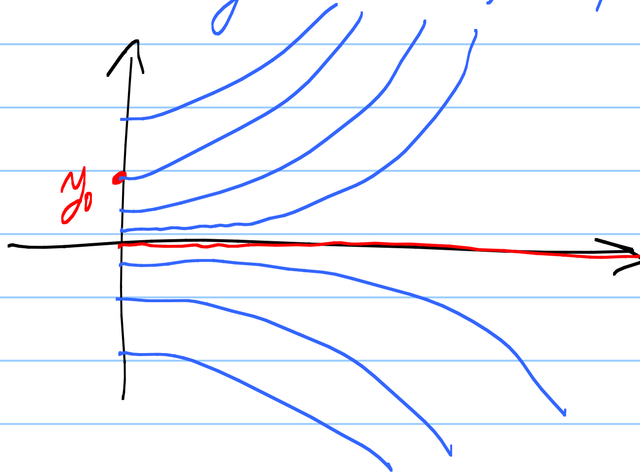
Gen^l Sol^n: $y = Ce^{kx}$, any C.

Plug in $x=0$ to get

$$y_0 = Ce^{k\cdot 0} = C$$

$$y = y_0 e^{kx}$$

$k>0$



Nice thing: sol^n's exist forever (all x). Linear equ!

$$\frac{dy}{dx} = k\,y \quad \leftarrow \text{first powers}$$

Non-linear ex:

$$\frac{dy}{dx} = y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} = 1 \qquad\qquad \int u^n\,du = \frac{1}{n+1}u^{n+1} + C$$

$$\frac{d}{dx}\left(-y^{-1}\right) = 1$$

$$-y^{-1} = \int 1\,dx = x + C$$

$$\boxed{y = \frac{-1}{x+C}}$$

Boom! at $x = -C$

Non-linear behaviors.

Rabbits in Australia:

$$\frac{dp}{dt} = \underbrace{k_1 p}_{\text{birth rate}} - k_2$$

death rate

EX:

$$\frac{dp}{dt} = 2p - 3$$

$$\frac{1}{2p-3}\frac{dp}{dt} = 1$$

Play around: $\frac{d}{dt} \ln|2p-3| = \frac{1}{2p-3} \frac{d}{dt}(2p-3)$

$$= \frac{1}{2p-3}\, 2\, \frac{dp}{dt}$$

$$\frac{1}{2}\frac{d}{dt} \ln|2p-3| = 1$$

$$\frac{1}{2}\ln|2p-3| = \int 1\,dt = t + C$$

$$e^{\ln|2p-3|} = e^{2t + 2C}$$

$$|2p-3| = e^{2t} e^{2C}$$

$$2p-3 = \underbrace{\pm e^{2c}}_{\tilde{C}} e^{2t}$$

$$p = \underbrace{\frac{\tilde{C}}{2}}_{\tilde{\tilde{C}}} e^{2t} + \frac{3}{2}$$

$\leftarrow$ call it C instead

Sol^n: $$p = \frac{3}{2} + Ce^{2t}$$

Plug in $t=0$:

$$p_0 = \frac{3}{2} + Ce^{2\cdot 0}$$

$$\boxed{C = p_0 - \frac{3}{2}}$$

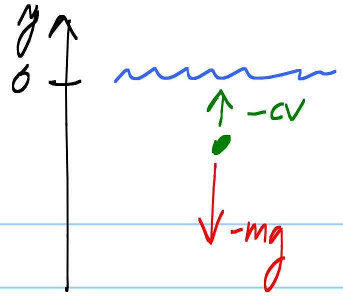
Sol^n: $$p = \frac{3}{2} + \left(p_0 - \frac{3}{2}\right)e^{2t}$$

Rock in ocean: $F = ma$

$$-mg - cv = m\frac{dv}{dt}$$ $\leftarrow$ Same eq^n!

Book: $$\frac{dy}{dt} = ay - b$$

$$y = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$$



$$m\frac{dv}{dt} = -mg - cv$$

$$\frac{dv}{dt} = -\frac{c}{m}v - g$$

Get $$v = \frac{-mg}{c} + \underbrace{\left(v_0 + \frac{mg}{c}\right)e^{-\frac{c}{m}t}}_{\text{fizzles out!}}$$

terminal velocity!

Drop rock: (from rest). $\boxed{v_0 = 0}$

$$\lim_{t\to\infty} v(t) = \frac{-mg}{c}$$ ← c = coeff of friction

Exciting! Can get y now!

$$y = \int \frac{dy}{dt}\,dt = \int v\,dt = \int \frac{-mg}{c} + \frac{mg}{c}e^{-\frac{c}{m}t}\,dt$$

$$= \frac{-mg}{c}t + \frac{mg}{c}\left(\frac{1}{\left(\frac{-c}{m}\right)}e^{-\frac{c}{m}t}\right) + K$$

$$\int e^{kt}\,dt = \frac{1}{k}e^{kt} + C$$

Know $y(0) = y_0 = 0$ ← surface

Plug $t=0$: $$0 = \frac{-mg}{c}\cdot 0 - \frac{m^2 g}{c^2}e^{0} + K$$

Get $$\boxed{K = \frac{m^2 g}{c^2}}$$

$$y = -\frac{mg}{c}t - \frac{m^2 g}{c^2} e^{-\frac{c}{m}t} + \frac{m^2 g}{c^2}$$

Gradescope: Single PDF's for each prob.