Lecture 3 1.3 Linear vs Nonlinear, examples

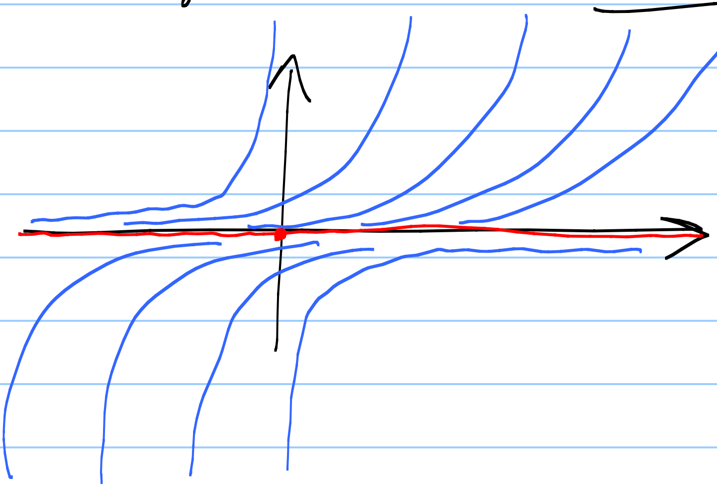
Linear ex: $\frac{dy}{dx} = ky$ $y = Ce^{kx}$ ← good for all x

Non-linear ex: $\frac{dy}{dx} = y^2$

Aha! Might divide by zero here! → $\frac{1}{y^2}\frac{dy}{dx} = 1$

Chain rule → $\frac{d}{dx}\left[-\frac{1}{y}\right] = 1$

$-\frac{1}{y} = x + C$ $\boxed{y = \frac{-1}{x+C}}$



← $y \equiv 0$ is a sol'n missing in formula

Linear ODE:

$$(x^{10} + \sin x)\frac{d^2y}{dx^2} + (\sqrt{x} + e^x)\frac{dy}{dx} + \left(x^7 + \frac{1}{x}\right)y = (x^2 + \tan x)$$

$= 0$ ← homogeneous

$$A_n(x)\frac{d^ny}{dx^n} + A_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_1(x)\frac{dy}{dx} + A_0(x)y = B(x)$$

Standard form: Divide by $A_n(x)$

$$1 \cdot \frac{d^ny}{dx^n} + \cdots = 0$$

EX:
$$x^2 \frac{dy}{dx} + 2xy = x^3$$

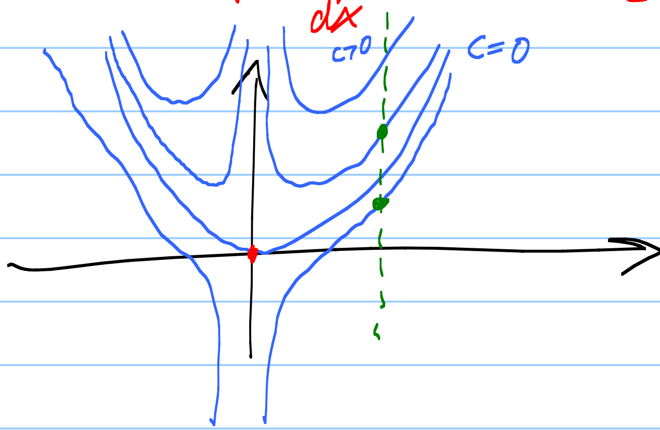
Aha! $\frac{d}{dx}\left[x^2 y\right] = x^2 \frac{dy}{dx} + 2xy$ ✓

So $x^2 y = \int x^3\,dx = \frac{1}{4}x^4 + C$

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

↑ boom! at $x=0$.

Zeroes of the coef of $\frac{dy}{dx}$ can cause problems.



EX:
$$\frac{dy}{dx} = 2y - 3$$

$$\frac{dy}{dx} - 2y = -3$$

Big idea: Multiply eqn by an "integrating factor" $u(x)$

$$\underbrace{u(x)\frac{dy}{dx} - 2u(x)\,y}_{} = -3u(x)$$

Choose u so that this $= \frac{d}{dx}\left[\text{product}\right]$

$$\overset{\text{want}}{=} \frac{d}{dx}\left[uy\right] = u\frac{dy}{dx} + \frac{du}{dx}\,y$$

Aha! If only we had $\boxed{\frac{du}{dx} = -2u}$ ← exponential decay prob

$$u = Ce^{-2x}$$

Just want <u>one</u> u that works. Take $\boxed{C=1}$

<u>Step 1</u>: Get u. ✓ $\boxed{u(x) = e^{2x}}$

<u>Step 2</u>: Mult eqn by u:

$$\underbrace{e^{-2x}\frac{dy}{dx} - 2e^{-2x}y}_{\frac{d}{dx}\left[e^{-2x}y\right]} = -3e^{-2x}$$

u

<u>Step 3: Integrate</u>

$$e^{-2x}y = \int -3e^{-2x}\,dx$$

$$= -3\frac{1}{(-2)}e^{-2x} + C$$

<u>Step 4</u>: Solve for y:

$$y = \frac{3}{2} + \frac{C}{e^{-2x}}$$

$$\boxed{y = \frac{3}{2} + Ce^{2x}}$$ Gen^l Sol^n

Find the sol^n with $y(0) = 2$ ← Initial value problem

$$y(0) = \frac{3}{2} + C\underbrace{e^{2\cdot 0}}_{1} \overset{\text{want}}{=} 2$$

$$C = 2 - \frac{3}{2} = \frac{1}{2}$$

Get $\boxed{y = \frac{3}{2} + \frac{1}{2}e^{2x}}$

Linear eqns. Heavenly

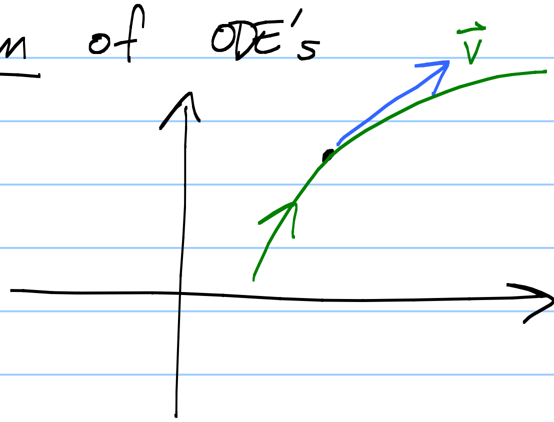
Non-linear eqns look like this. (Solve for highest order derivative)

$$\underbrace{\frac{d^n y}{dx^n} = f\left(\frac{d^{n-1}y}{dx^{n-1}}, \dots, \frac{dy}{dx}, y, x\right)}$$

standard form

O.D.E. "Ordinary differential eqn" (y is a fcn of one variable)

EX: System of ODE's



$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

Know wind direction:

$$\begin{cases} \dfrac{dx}{dt} = f(x,y,t) \\ \dfrac{dy}{dt} = g(x,y,t) \end{cases}$$

PDE's "Partial differential equs"

EX: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$ Laplace's equ

$\Delta u = c\frac{\partial u}{\partial t}$ Heat equ

$\Delta u = c\frac{\partial^2 u}{\partial t^2}$ Wave equ

How to solve linear ODEs:

$$A(x)\frac{dy}{dx} + B(x)\,y = G(x)$$

Step 1: Divide by $A(x)$ to put in Standard form

Get $\frac{dy}{dx} + p(x)\,y = g(x)$ $\left(\begin{array}{l} p = \frac{B}{A} \\ g = \frac{G}{A} \end{array}\right)$

Step 2: Mult by $u(x)$:

$$\underbrace{u\frac{dy}{dx} + up\,y}\ = ug$$

need these to be equal

Wish: $\frac{d}{dx}[u\,y] = u\frac{dy}{dx} + \frac{du}{dx}y$

Need $\boxed{\frac{du}{dx} = p(x)\,u(x)}$

$$\frac{1}{u}\frac{du}{dx} = p(x)$$

$$\frac{d}{dx} \ln|u| = p(x)$$

$$\ln|u| = \int p(x)\,dx + C$$

$$|u| = e^{\int p(x)\,dx + C} = e^{C} e^{\int p(x)\,dx}$$

$$u = \underbrace{(\pm e^{C})}_{K} e^{\int p(x)\,dx}$$

Just want one u that works. Take $K=1$.

Big formula: $\boxed{u = e^{\int p(x)\,dx}}$

Step 3: Use it!

$$\frac{dy}{dx} + p(x)y = g(x) \leftarrow \text{Standard form!}$$

$$\underbrace{u\left(\frac{dy}{dx} + p(x)y\right)}_{\frac{d}{dx}[uy]} = ug$$

Integrate: $uy = \int ug\,dx + C$

Step 4: Solve for y

$$y = \frac{1}{u(x)} \int u(x) g(x) \, dx + \frac{C}{u(x)}$$

Don't memorize. Memorize u formula

$$\boxed{u = e^{\int p \, dx}}.$$

Use it as above.