Lecture 3 — Linear equations

HW 1,2,3 due tonight in MyLab

HWK 2w, 3w due Wed in Gradescope

$$A(x)\frac{dy}{dx} + B(x)y = Q(x)$$

Step 1: Divide by $A(x)$ to put in Standard Form.

$$\frac{dy}{dx} + p(x)y = g(x) \qquad p(x) = \frac{B(x)}{A(x)}$$

Step 2: Calculate int. factor

$$u = e^{\int p(x)\,dx}$$

Step 3: Multiply Stand. Form eqn by u:

$$\underbrace{u\frac{dy}{dx} + upy}_{\frac{d}{dx}[uy]} = ug$$

Step 4: Integrate:

$$uy = \int ug\,dx + C$$

Step 5: Solve for y

$$y = \frac{1}{u}\int ug\,dx + \frac{C}{u}$$

Very cool thing: Can calculate integrals via the Fund Thm Calc:

$$u(x) = e^{\int_{x_0}^{x} p(t)\,dt}$$

I.V.P. (Initial Value Problem)

$$y(x_0) = y_0$$

$$u(x_0) = e^{\int_{x_0}^{x_0}\cdots} = e^0 = 1$$

$$y(x) = \frac{1}{u(x)} \int_{x_0}^{x} u(t)g(t)\,dt + \frac{C}{u(x)}$$

$$y(x_0) = \frac{1}{1}\cdot \underbrace{\int_{x_0}^{x_0} \cdots dt}_{0} + \frac{C}{1} = C \overset{\text{want}}{=} y_0$$

Wow! If $p(x)$ and $g(x)$ are by data, Can use Simpson's Rule to compute areas!

Philosophical fact: If $p(x)$ and $g(x)$ are continuous fcns, then $y(x)$ is continuosly diff'ble. The solution exists and is unique!

EX: $$2\frac{dy}{dx} + y = e^{3x}$$

1: $$\frac{dy}{dx} + \frac{1}{2} y = \frac{1}{2} e^{3x}$$ Standard Form!

$p(x) = \frac{1}{2}$

2: $$u = e^{\int p(x)dx} = e^{\int \frac{1}{2} dx} = e^{x/2}$$ No $+C$ here. Just need one u.

3: $$\underbrace{e^{x/2}\left(\frac{dy}{dx} + \frac{1}{2} y\right)}_{\frac{d}{dx}\left[e^{x/2}\, y\right]} = e^{x/2}\left(\frac{1}{2} e^{3x}\right)$$

$e^{x/2}$ ← u

4: $$e^{x/2} y = \frac{1}{2}\int e^{(7/2)x}dx = \frac{1}{2}\,\frac{1}{\left(\frac{7}{2}\right)}\,e^{(7/2)x} + C$$

5: $\boxed{y = \frac{1}{7} e^{3x} + C e^{-x/2}}$

EX: $\frac{dy}{dx} - (\text{Tan } x) y = 8 \text{ Sin}^3 x$

Standard Form. ✓

$\boxed{a \ln b = \ln b^a}$

$u(x) = e^{\int (-\text{Tan } x) dx} = e^{-\ln|\text{Sec } x|} = e^{\ln|\text{Sec } x|^{-1}}$

$= |\text{Sec } x|^{-1} = |\text{Cos } x|$

$\text{Sec} = \frac{1}{\text{Cos}}$

$= \pm \text{ Cos } x$ ← pick + one.

Take $\boxed{u(x) = \text{Cos } x}$

$\underbrace{\text{Cos } x \left(\frac{dy}{dx} + (-\underbrace{\text{Tan } x}_{\frac{\text{Sin}}{\text{Cos}}}) y \right)}_{\frac{d}{dx}[\text{Cos } x \; y]} = \text{Cos } x \, 8 \text{ Sin}^3 x$

$\frac{d}{dx}[\underset{u}{\text{Cos } x} \; y]$

$\text{Cos } x \; y = 8 \int \underset{v = \text{Sin } x}{\text{Sin}^3 x} \; \underbrace{\text{Cos } x \; dx}_{dv = \text{Cos } x \, dx}$ ✓

$$= 8 \cdot \frac{1}{4} v^4 + C$$

$$\cos x \; y = 2 \sin^4 x + C$$

$$y = 2 \frac{\sin^4 x}{\cos x} + \frac{C}{\cos x}$$

$$= 2 \frac{\sin x}{\cos x} \sin^3 x + C \sec x$$

$$\boxed{y = 2 \tan x \sin^3 x + C \sec x}$$

Hmmm. $\sec x$ is nasty at $\frac{\pi}{2}$!

$$\frac{dy}{dx} + \underbrace{(-\tan x)} y = 8 \sin^3 x$$

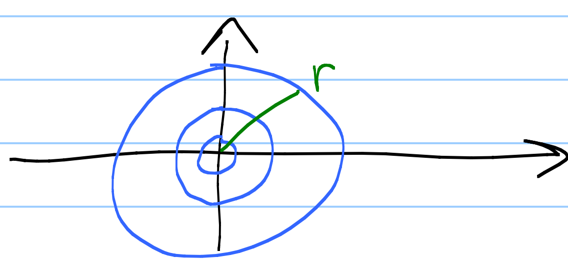
Aha! $p(x)$ is nasty at $\frac{\pi}{2}$

Fact: $A(x) \frac{dy}{dx} + B(x) y = Q(x)$

Step 1: Divide by A to get p and g.

Zeroes of $A(x)$ are dangerous!

Problem: Find an O.D.E. that has solutions:



Circles

$$x^2 + y^2 = r^2 = C, \text{ a constant}$$

Implicit differentiation!

$$\underbrace{\frac{d}{dx}(x^2)}_{2x} + \underbrace{\frac{d}{dx}(y^2)}_{(2y)\frac{dy}{dx}} = \frac{d}{dx}(C) = 0$$

Cancel 2's. $\quad x + y\frac{dy}{dx} = 0$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

<u>Separable first order ODEs:</u> $\quad \frac{dy}{dx} = \frac{f(x)}{g(y)}$

<u>Method</u>:

$$g(y)\,dy = f(x)\,dx \leftarrow \text{Separate } x\text{'s}, y\text{'s}$$

Integrate $\quad \int g(y)\,dy = \int f(x)\,dx + C$

Defines solutions $y(x)$ to ODE <u>implicitly</u>.

Solve for y if you can.

Does this really work?

$$\underbrace{\int g(y)\,dy}_{G(y)} = \underbrace{\int f(x)\,dx}_{F(x)} + C$$

$$G(y) - F(x) = C \leftarrow \text{defines } y(x)$$

Implicit diff'tion: $\underbrace{\frac{d}{dx} G(y)}_{\underbrace{G'(y)}_{g(y)} \frac{dy}{dx}} - \underbrace{\frac{d}{dx} F(x)}_{f(x)} = \frac{d}{dx} C = 0$

$$g(y)\frac{dy}{dx} - f(x) = 0$$

$$\frac{dy}{dx} = \frac{f(x)}{g(x)} \quad \checkmark$$