Lecture 5 Separable, linear, E & U

HW04 due MyLab
HW02w, HW03w due Gradescope

Separable equ: $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$

$$\int g(y)\,dy = \int f(x)\,dx$$

$$G(y) = F(x) + C$$ ← defines sol^n $y(x)$ implicitly.

EX: $\dfrac{dy}{dx} = \dfrac{y}{x} = \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{y}\right)}$ ← $f(x)$, ← $g(y)$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

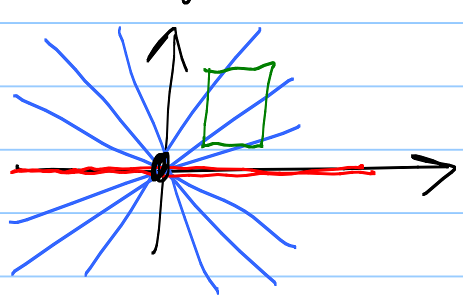
$$e^{\ln|y|} = e^{\ln|x| + C}$$

$e^{a+b} = e^a e^b$

$$|y| = |x|\,e^{C}$$

$$y = \pm e^{C}|x| = \underbrace{(\mp e^{C})}_{K}\, x$$

$y = Kx$



$$\frac{dy}{dx} = \frac{y}{x}$$

EX: $\frac{dy}{dx} = e^{x+y} = e^x e^y$

$$\int \frac{1}{e^y}\, dy = \int e^x \, dx$$

e^{-y}

$$\frac{1}{(-1)} e^{(-1)y} = e^x + C$$

$$e^{-y} = -e^x - C$$

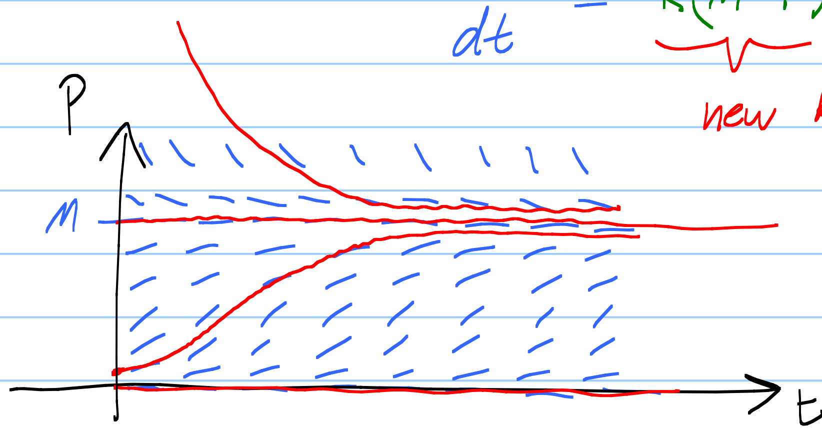
$$-y = \ln(\underbrace{-C}_{K} - e^x)$$

$$y = -\ln(K - e^x)$$

See $K>0$. Boom when $e^x = K$. $x = \ln K$.

Logistics eqn: $\frac{dP}{dt} = KP$ ← no limits to growth

$$\frac{dP}{dt} = \underbrace{K(M-P)}_{\text{new } K \text{ that depends on } P} P$$



$$\frac{dP}{dt} = k(M-P)P$$

$$\int \frac{dP}{(M-P)P} = \int k\,dt = kt + C$$

Partial fractions! $\frac{1}{(M-P)P} = \frac{A}{M-P} + \frac{B}{P}$

Mult by denom $(M-P)P$:

$$\underline{1} = AP + B(M-P)$$

$$0 \cdot P + 1 = \underbrace{(A-B)}_{0}P + \underbrace{(BM)}_{1}$$

$BM = 1$ $\boxed{B = \frac{1}{M}}$

$A - B = 0$ $A = B = \frac{1}{M}$ $\boxed{A = \frac{1}{M}}$

$$\int \frac{dP}{(M-P)P} = \int \left(\frac{1/M}{M-P} + \frac{1/M}{P} \right) dP = kt + C$$

$$= \frac{1}{M}\left(-\ln|M-P| + \ln|P| \right) = kt + C$$

$$\ln|P| - \ln|M-P| = kMt + \underbrace{CM}_{C}$$

$\boxed{\ln a - \ln b = \ln \frac{a}{b}}$

$$e^{\ln\left|\frac{P}{M-P}\right|} = e^{kMt + C}$$

$$\left|\frac{P}{M-P}\right| = e^{kMt}e^{C}$$

$$\frac{P}{M-P} = \underbrace{\left(\pm e^{C}\right)}_{C} e^{kMt}$$

$$P = Ce^{kMt}(M-P)$$

$$P = CMe^{kMt} - Ce^{kMt}P$$

$$P\left(1 + Ce^{kMt}\right) = CMe^{kMt}$$

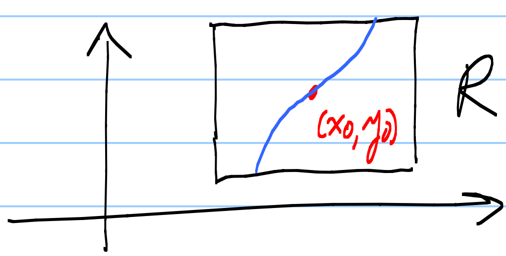
$$P = \frac{CMe^{kMt}}{1 + Ce^{kMt}} \qquad \text{Done!}$$

Easy to see $\lim_{t\to\infty} P(t) = M$.

Existence and Uniqueness Theorem for gen' first order ODE

$$\frac{dy}{dx} = f(x,y) \quad \text{with} \quad y(x_0) = y_0$$

Initial value problem



sol^n $y(x)$

Suppose 1) $f(x,y)$ is continuous on R, and

2) $\frac{\partial f}{\partial y}$ is also continuous on R

Then there exists a sol^n passing through (x_0, y_0)

[i.e., $y(x_0) = y_0$] and it is unique, and

it is continuously diff'ble as long as it stays in R.

Note: Can only say we get a sol^n on some interval

$(x_0 - h, x_0 + h)$ with $h > 0$.

EX: $\frac{dy}{dx} = y^2$

$f(x,y) = y^2$ cont. ✓

$\frac{\partial f}{\partial y} = 2y$ cont. ✓

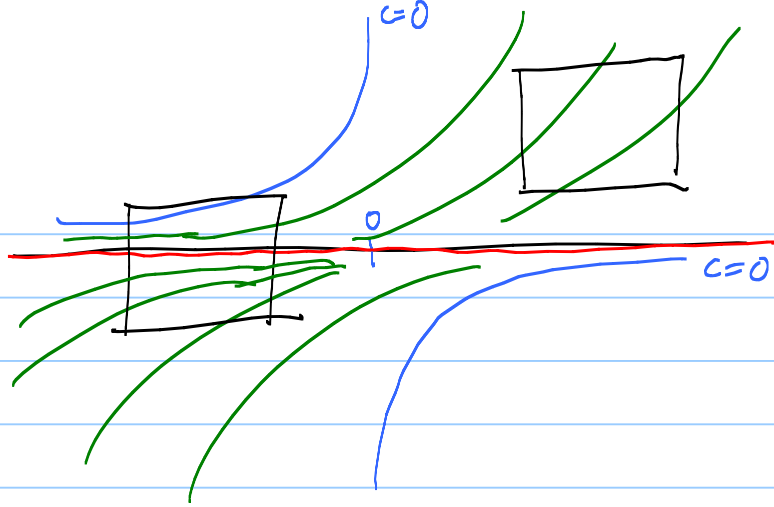
$$\frac{1}{y^2}\, dy = dx$$

$$\int y^{-2}\, dy = \int dx$$

$$\frac{1}{(-2)+1}\, y^{(-2)+1} = x + C$$

$$-\frac{1}{y} = x + C$$

$$y = \frac{-1}{x+C}$$



EX: $\dfrac{dy}{dx} = 2\sqrt{|y|}$ ← cont. ✓

$f(x,y) = 2y^{1/2}$ $\quad y > 0$

$\dfrac{\partial f}{\partial y} = 2\,\tfrac{1}{2}\,y^{\frac{1}{2}-1} = \dfrac{1}{\sqrt{y}}$ ← ouch! when $y=0$

Fact: If $f(x,y)$ is only continuous and $\frac{\partial f}{\partial y}$ is bad, still get existence, but we might lose uniqueness.